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Manuscripts should be typed on 8½″ x 11″ paper, double-spaced with 1″ margins. We prefer that, in technical papers, the usual introduction be preceded by a *Foreword* which states in simple terms what the paper is about.

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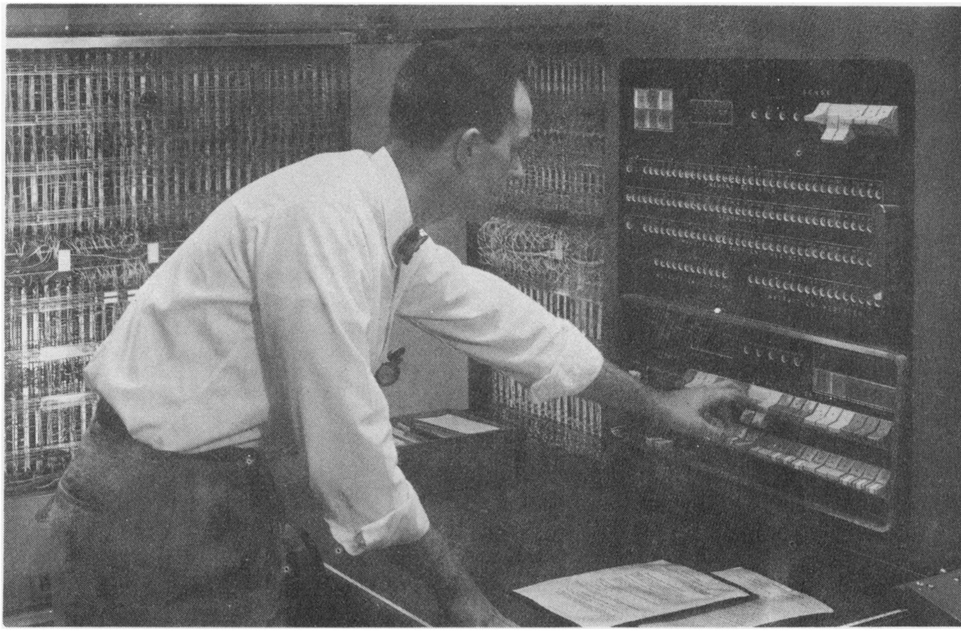
CONTENTS

The announced paper by a blind boy on "How I Learned Mathematics Without Brail" *will appear in the May-June issue.*

Editors Note:

Several inquires about what constitutes a "popular" article and the receipt of several papers proposed as "popular" that are merely **repetitions** of elementary mathematics, mostly collegiate, prompt me to offer a definition of "popular" which at least qualifies articles for the MATHEMATICS MAGAZINE.

A popular mathematics article is one which any intelligent person can read understandingly. This implies that it is free from technical symbols, except possibly those that occur in arithmetic, and that its statements do not presuppose specialization in mathematics or any other studies.



Mathematical Analyst *Keith Kersery loads jet transport flutter problem into one of Lockheed's two 701's. On order: two 704's to help keep Lockheed in forefront of numerical analysis and production control data processing.*

704's and 701's speed Lockheed research in numerical analysis

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SYMMETRICAL TYPES OF CONVEX REGIONS

Andrew Sobczyk

Foreword

A region C in the plane, or in vector space of any number of dimensions, is *convex* if whenever P, Q are points of C, the line segment PQ is entirely contained in C. A region S is *starlike* if there is a point θ of S, such that for every direction, the intersection with S of the ray or half-line from θ in the direction is either the entire ray, a segment with θ as one end-point, or only the point θ (as may be the case if θ is on the boundary of S). To facilitate analytic rather than synthetic study of symmetry, in this paper so-called gauge-functions are associated with convex and starlike regions.

For examples, the interior of a triangle is an open convex region; the interior plus the edges and vertices is a closed convex region. The interior of any non-convex quadrilateral, not having intersecting sides, is a starlike region. The gauge-function for an ellipse C is $p(x) = [(x_1/a_1)^2 + (x_2/a_2)^2]^{\frac{1}{2}}$; C is the set of all points x for which $p(x) \leq 1$.

A plane region R has *involutory symmetry* in the origin if for every point $x = (x_1, x_2)$ of R, the point $(-x_1, -x_2)$ is also a point of R. Unlike a circle, ellipse, or parallelogram, a triangle does not possess involutory symmetry in a point.

An ellipsoid with two equal semi-axes is generated by rotation of an ellipse about the third axis, and therefore has complete rotational symmetry about the axis. In this paper, we explore the possible types and interrelations of rotational and involutory symmetries which may be possessed by regions in n dimensional space.

Plane and solid analytic geometry, elementary group theory, and some acquaintance with matrix algebra, or study of reference [2], are sufficient for understanding most of the paper. For full appreciation of section 2, however, it would be desirable to have a knowledge of the elements of the theory of normed vector spaces.

1. Introduction

Section 2 is devoted to gauge-functions, which are functions on n dimensional vector space E_n to the real half-line $[0,\infty]$, "closed" or "compactified" by addition of an ideal point $+\infty$. It is shown that any convex or starlike region has an associated gauge-function $p(x)$ of suitable nature, and conversely. (If the region is open, or if the origin is interior to the region, the associated gauge-function does not assume the value $+\infty$.) It is proved that any non-negative function $p(x)$ which has any two of the properties positive homogeneity, subadditivity, convexity, must have all three properties, and is the gauge-function for a convex region.

In section 3, after recalling for the reader the geometric meaning of involutory symmetry in any subspace of E_n, it is proved that if a region C has such symmetry in each of a set of complementary linear subspaces $L_1,\ldots,L_k$ of E_n, then C is symmetric with respect to the origin (that is, the gauge-function for C is a Minkowski or Banach norm on E_n). Possible regions which have specified sections $C_1,\ldots,C_k$ in $L_1,\ldots,L_k$ are discussed; a region C is said to be of *standard type* with respect to $L_1,\ldots,L_k$ if the projections of C onto $L_1,\ldots,L_k$ coincide respectively with the sections of C in $L_1,\ldots,L_k$. Here an interesting question is the following: Is every convex region C in E_n, after a suitable translation, of standard type with respect to some choice of n one-dimensional subspaces $H_1,\ldots,H_n$ which span E_n? (The author has presented a preliminary report [3], which includes an affirmative partial answer to this question, to the American Mathematical Society.) It is also shown in section 3, by use of gauge-functions, that the origin θ is not a boundary point for any convex set C which is symmetric in θ, unless C is entirely contained in a linear subspace E_k of E_n, $k<n$.

Generation of rotationally symmetric regions is discussed in section 4, and it is shown that the gauge-function for any such region may be expressed in terms of lower dimensional gauge-functions. Also it is proved that if a region C is invariant under any sub-interval of rotations, then it is invariant under the rotations of the entire interval $0 \leq \phi \leq 2\pi$.

A region C which has involutory symmetry in complementary linear subspaces $L_1,\ldots,L_k$ is symmetric in the origin θ because the product of the corresponding involutions $V_1,\ldots,V_k$ is the negative identity $(-I)$; likewise C has symmetry in θ if C is invariant under any set of transformations whose product is $(-I)$; some facts about possible regions C which are invariant under various groups of transformations are derived in section 5. In section 6, it is shown that for any bounded convex or starlike region C with θ in its interior, any possible linear transformation T under which C may be invariant is such that there exists a choice of basis vectors for E_n, with respect to which T appears as an orthogonal transformation. It is also shown that if C is any region and T is any non-involutory orthogonal transformation, then there exists a region $J(C)$, not essentially different from C, such that TC equals $UJ(C)$, where U is an involution. Finally in Section 7, there is a discussion of so-called "composite" regions, that is, regions which correspond to lower dimensional regions, in an analogous way to the correspondence of rotationally symmetric regions to lower dimensional spheres.

2. Gauge-functions

If $x = (x_1, \ldots, x_n)$ is a point of E_n, where the coordinates are with respect to any particular basis, then for any real r, the multiple $rx = (rx_1, \ldots, rx_n)$ is also a point of E_n. For any set R of E_n, we may consider the set rR which consists of the multiples rx for all x in R.

For a convex or starlike region A, located so that the point θ, contained in or on the boundary of A (in case of convex A, θ may be any such point), is the origin of E_n, we define a *gauge-function* $p(x)$ on E_n to be the infimum (greatest lower bound) of values of $r > 0$ for which x is contained in rA; if x is in rA for no positive r, we define $p(x) = +\infty$. If A is unbounded in the direction x, the infimum is zero. We define also $p(\theta) = 0$. (Compare with [1].) It follows immediately from the definition that $p(x)$ is positive homogeneous; that is, that $p(ax) = ap(x)$ for all $a > 0$ (wi the understanding that $a \cdot (+\infty) = +\infty$ for all $a > 0$).

If $p(x)$ is the gauge-function for A, then evidently A contains all points x for which $p(x) < 1$, and is contained in the set for which $p(x) \leq 1$. If it is desired to consider regions A which do not contain all points for which $p(x) = 1$, let such points which are not in A be omitted from the domain of definition of $p(x)$. With this convention, A in any case is the set of x for which $p(x) \leq 1$.

Even if no points are omitted from the domain of definition of $p(x)$, the set A of all x for which $p(x) \leq 1$ may not be closed. This is shown by the example $p(x) = 0$ in an open angular sector of the plane, $p(\theta) = 0$, and except at θ, $p(x) = +\infty$ outside the sector.

For the main purposes of this paper, there is no loss of generality in assuming that the regions A considered are such that no points of E_n are required to be omitted from the domain of definition of $p(x)$. Therefore, throughout the rest of the paper, let it be understood that this assumption is made.

REMARK 2.0. *If p_1 and p_2 are the gauge-functions for two starlike or convex regions S_1, S_2, then it follows immediately from the definition of gauge-function that $S_1 \subseteq S_2$ if and only if $p_1(x) \geq p_2(x)$ for all x in E_n.*

Theorem 2.1. *Any positive homogeneous, non-negative function $p(x)$, with $p(\theta) = 0$, is the gauge-function for a starlike region A with respect to θ, and conversely any starlike region A has an associated positive homogeneous function $p(x)$ such that A is the set of all x for which $p(x) \leq 1$.*

Proof: The converse has been established above. Let A be the set of x for which $p(x) \leq 1$. If $x \neq \theta$ and $p(x) \leq 1$, then the segment $\{cx\}$, $0 \leq c \leq a$, where $p(ax) = 1$, is contained in A by positive homogeneity; in case $p(x) = 0$, the ray $\{cx\}$, $0 \leq c < \infty$, is contained in A; therefore A is starlike.

A function $q(x)$ on E_n is *subadditive* if $q(x+y) \leq q(x) + q(y)$ for all x, y in E_n. Let any positive homogeneous, non-negative, subadditive function $q(x)$, with $q(\theta) = 0$, be called a *pseudo-sub-norm*; and a *pseudo-semi-norm* in case $q(x) < +\infty$ for all x in E_n. Let the prefix *pseudo-* be dropped in case $q(x) > 0$ for all $x \neq \theta$. For convex regions we have the following theorem.

Theorem 2.2. *Any pseudo-sub-norm $q(x)$ is the gauge-function for a convex region C, with the origin θ interior to or on the boundary of C; and conversely, the gauge-function for any convex region C is a pseudo-sub-norm. A similar statement, with "pseudo-sub-norm" replaced by "pseudo-semi-norm," may be made, in case θ is interior to C. In both statements, the prefix "pseudo-" may be dropped in case C is bounded (in a Euclidean metric on E_n), except that a sub-norm $q(x)$ which is not a semi-norm ($q(x) = +\infty$ for some x of E_n) may correspond to an unbounded C.*

Proof: We first establish the converse statements, by proving that the gauge-function $p(x)$ for C, as defined above, has the required properties. For θ and any x in E_n, we have $p(\theta + x) = p(x) \leq p(\theta) + p(x)$ since by definition $p(\theta) = 0$. If x, y of E_n are both different from θ, we have the following possible cases: (i) $p(x) = p(y) = 0$; (ii) $+\infty > p(x) > 0$, $+\infty > p(y) \geq 0$; (iii) $+\infty = p(x)$, $+\infty \geq p(y) \geq 0$. In case (i), for arbitrarily small ϵ, there exist x', y' in C such that $x = \epsilon x'$, $y = \epsilon y'$, and by convexity of C and positive homogeneity of p, we have $p(\tfrac{1}{2}x' + \tfrac{1}{2}y') \leq 1$, $p(x' + y') \leq 2$, $p(x+y) = \epsilon p(x' + y') \leq 2\epsilon$; therefore $p(x + y) = 0 \leq p(x) + p(y)$, and p is subadditive as required. In case (ii), let $p(x) = k$, $x' = x/k$, $y' = y/c$, where $c = p(y)$ if $p(y) > 0$; otherwise c may be positive and arbitrarily small. Then $p(x') = 1$, $p(y') = 1$ or 0, $kx'/(k + c) + cy'/(k + c) = (x + y)/(k + c)$, and by convexity of C and positive homogeneity of p, we have $p(kx' + cy')/(k + c) = p[(x + y)/(k + c)] \leq 1$, or $p(x + y) \leq k + c$. Thus, we have either $p(x + y) \leq p(x) + p(y)$, or $p(x + y) \leq p(x) + p(y) + c$ for arbitrarily small c, and p is subadditive. In case (iii), since p is non-negative and in particular $p(y) \neq -\infty$, we have $p(x) + p(y) = +\infty \geq p(x+y)$, for all x, y in E_n. If θ is interior to C, then every point $y \neq \theta$ of E_n is of the form cx, with c positive and x in C; therefore $p(y) < +\infty$, the cases where $p = +\infty$ do not arise, and always $p < +\infty$, so p is a pseudo-semi-norm. If C is bounded, then for each $y \neq \theta$ there will exist a positive r such that y is not in rC, and the cases where $p = 0$ do not occur; that is, p is a sub-norm or semi-norm.

Verification of the direct statements of the theorem is immediate: If $q(x)$ is a pseudo-sub-norm, then the region C defined by $q(x) \leq 1$ is convex, and has $q(x)$ for its gauge-function. For if x, y are in C, then by positive homogeneity and subadditivity, $q[ex + (1 - e)y] \leq$

$eq(x) + (1-e)\ q(y) \leq 1$ for any e in $0 \leq e \leq 1$, since by hypothesis $q(x) \leq 1$, $q(y) \leq 1$. If $q(x)$ is a pseudo-semi-norm, then $q(x) < +\infty$ for all x of E; that is, x is in rC for some finite r, and θ must be interior to C.

If $q(x)$ is a semi-norm, $q(x) < +\infty$ and $q(x) > 0$ for all $x \neq \theta$. The function $r(x) = q(x) + q(-x)$ has the property $r(bx) = |b|\ r(x)$ for all real b, since $r(bx) = q(bx) + q(-bx) = r(-bx)$; therefore $r(x)$ is a norm. Similarly $s(x) = max\ [q(x), q(-x)]$ is a norm. It is well-known that any norm on E_n is isomorphic with the Euclidean norm; that is, any norm $r(x)$ is bounded and bounded from zero on the Euclidean unit sphere $\|x\| = 1$. By compactness of the unit sphere, any infinite sequence of points on the sphere has a subsequence $\{x^i\}$ which is convergent to a point x of the sphere. Assuming that $C = \{x|q(x) \leq 1\}$ is unbounded, that is, that $q(x^i) \to 0$, we have $r(x^i) = q(x^i) + q(-x^i) \to r(x)$ since r is a norm, or $r(x) = q(x) + q(-x) = d = \lim q(-x^i)$. Since by hypothesis $q(x) > 0$, $q(-x) > 0$, we have $0 < q(x) < d$, $0 < q(-x) < d$. But since $q(x^i) \to 0$ and $s(x)$ is a norm, $s(x^i) = max\ [q(x^i), q(-x^i)] \to \lim q(-x^i) = d = s(x) = max\ [q(x), q(-x)]$, so either $q(x)$ or $q(-x)$ is equal to d. From the contradiction it follows that C must be bounded.

Finally, the exception at the end of the statement of Theorem 2.2 is established by the following example: For points $x = (x_1, \ldots, x_n)$ of E_n, define $q(x) = x_n/b$, $b > 0$, for $x_n > 0$, $q(\theta) = 0$, and $q(x) = +\infty$ for $x \neq \theta$, $x_n \leq 0$. Then q is subadditive, positive homogeneous, and $q(x) > 0$ for all $x \neq \theta$; that is, q is a sub-norm, but $C = \{x|q(x) \leq 1\}$ consists of the unbounded slab $0 < x_n \leq b$ and the boundary point θ.

A function $q(x)$, on E_n or a convex subset of E_n, to the real number line compactified by the addition of both $-\infty$ and $+\infty$, is *convex* in case $q[ex + (1-e)y] \leq eq(x) + (1-e)\ q(y)$, for all x,y in E_n and e in $0 \leq e \leq 1$. If q is defined only on a convex proper subset of E_n, define $q(x) = +\infty$ through the remainder of E_n; this does not change the convexity of q. The set C_k of all x for which $q(x) \leq k$, for any constant k in $[-\infty, +\infty]$ is either empty or convex. For if x,y are in C_k, we have $q[ex + (1-e)y] \leq ek + (1-e)k = k$; that is, $ex + (1-e)y$ is in C_k, and C_k is convex by definition. Similarly $C_\infty = \{x|q(x) < \infty\}$ is convex. Thus, there is associated with every convex function $q(x)$, a one-parameter family of convex sets $\{C_k\}$, together with a convex set C_∞ which includes C_k for every finite k.

If $f(t)$ is any monotone non-decreasing, convex function on $[-\infty,\infty]$ to $[0,\infty]$, for example the exponential function, and if $q(x)$ is convex on E_n, then $f[q(x)]$ is convex. For we have $f\{q[ex + (1-e)y]\} \leq f\{eq(x) + (1-e)q(y)\} \leq ef[q(x)] + (1-e)f[q(y)]$. Therefore, to every convex function with range in $[-\infty,\infty]$, there correspond convex functions with range in $[0,\infty]$, which have the same associated family of convex sets in E_n.

If p is a convex function with range included in $[0,\infty]$, then unless $p(x) = +\infty$ for all x in E_n, there exists a point w in E_n such that $p(w) < \infty$. The convex function $r(x) = p(x + w)$ then has $r(\theta) < \infty$, and the associated family of $r(x)$ is a translated associated family of $p(x)$. In case the infimum h of p is attained at a point of E_n, this point may be taken as w, and $r(x) = p(x + w) - h$ will be a non-negative convex function, with $r(\theta) = 0$, having essentially the same associated one-parameter family of convex sets as an arbitrary convex function q with range in $[-\infty,\infty]$ which attains its infimum. If the arbitrary function q does not attain its infimum, then the equivalent (in the sens v of having the same associated family) non-negative function r must have $r(\theta) > 0$, and $r(x) > 0$ for all x in E_n, but $\inf r(x) = 0$.

In case $r(x)$ is a gauge-function for a region C, the region $r(x) \leq k$, for each k, is similar or homothetic to the region $C = \{x | r(x) \leq 1\}$. This is not the case for the family of convex sets associated with a general convex function. We have, however, the following theorem relating non-negative convex functions with $r(\theta) = 0$, and gauge-functions.

Theorem 2.3. *For functions $r(x)$ with domain E_n, and range included in $[0,\infty]$, with $r(\theta) = 0$, any two of the properties subadditivity, positive homogeneity, convexity, imply the third.*

Proof: Suppose first that $r(x)$ is positive homogeneous and subadditive. Then respectively by subadditivity and positive homogeneity, $r[ex + (1-e)y] \leq r(ex) + r[(1-e)y] = er(x) + (1-e)r(y)$ for $0 \leq e \leq 1$, so $r(x)$ is convex.

Suppose $r(x)$ is positive homogeneous and convex. Then respectively by positive homogeneity and convexity with $e = \frac{1}{2}$, $r(x + y) = 2r(\frac{1}{2}x + \frac{1}{2}y) \leq 2[\frac{1}{2}r(x) + \frac{1}{2}r(y)] = r(x) + r(y)$, so $r(x)$ is subadditive.

Suppose $r(x)$ is subadditive and convex. For $b > 1$ between integers n and $n + 1$, let $b = n + d = (1-d)n + d(n + 1)$. Then by subadditivity $r(nx) \leq nr(x)$, $r[(n + 1)x] \leq (n + 1)r(x)$, and by convexity $r(bx) = r[(1-d)nx + d(n + 1)x] \leq (1-d)r(nx) + dr[(n + 1)x] \leq (1-d)nr(x) + d(n + 1)r(x) = [(1-d)n + d(n+1)]r(x) = br(x)$. If $a = 1/b$, $y = bx$, then by the preceding inequality, $r(y) \leq br(y/b)$, $[r(y)/b] \leq r(y/b)$, or $ar(y) \leq r(ay)$ for $0 < a < 1$. On the other hand, $r(ay) = r[(1-a)\theta + ay] \leq (1-a)r(\theta) + ar(y) = ar(y)$, by convexity. Therefore, $r(ay) = ar(y)$. For $b > 1$, let $x = (y/b) = ay$; then $r(x) = (1/b)r(bx)$, or $r(bx) = br(x)$ for all x in E_n and $b > 1$. Therefore $r(x)$ is positive homogeneous.

3. Involutory Symmetry in Complementary Subspaces.

It follows immediately from the definition of gauge-function, that a convex or starlike set K is symmetrical in the origin θ, if and only if its gauge-function $q(x)$ has the property $q(-x) = q(x)$ for all x in E_n. Let any gauge-function with the latter property be called *symmetric*. We have the following theorem concerning symmetric gauge-functions.

Theorem 3.1. *A symmetric pseudo-semi-norm $p(x)$ is a pseudo-norm. For any symmetric pseudo-sub-norm $q(x)$, if L is the smallest linear subspace of E_n which contains the subset of E_n where $q(x) < \infty$, then q is a pseudo-norm on L, and $q(x) = \infty$ for all x in the set $(E_n - L)$.*

Proof: If $p(x)$ is a symmetric pseudo-semi-norm, then by hypothesis $p(-x) = p(x)$, and for $a > 0$, $p(-ax) = ap(-x) = ap(x)$, so $p(x)$ has the property of homogeneity, which is required in order that it be a pseudo-norm.

Let L be the smallest linear subspace which contains the set $C = \{x | q(x) < \infty\}$. If L contains a one-dimensional subspace E_1, then by symmetry if $x \neq \theta$ is in C and in E_1, $q(-x) = q(x) < \infty$, so $E_1 \subset C$. If C contains a linear subspace E_{k-1}, of dimension $(k-1)$, and an element y not in E_{k-1}, then by symmetry $q(-y) = q(y)$, and by sub-additivity and positive homogeneity, $q(x \pm cy) \leq q(x) + cq(\pm y)$ for any $c > 0$ and x in E_{k-1}, so q is finite in the linear subspace E_k spanned by E_{k-1} and y. Therefore $L = C$, and q is a pseudo-norm in L.

Theorem 3.1 suggests the following definition. A convex region or set K in E_n is a convex *body*, in case the gauge-function $q(x)$ for K has the property that $C = \{x | q(x) < \infty\}$ is not contained in any E_k for $k < n$. The geometric meaning of Theorem 3.1 is that the origin θ is not a boundary point of any convex body K which is symmetrical in θ.

Any transformation U of E_n onto E_n, such that $U^2 = I$, where I is the identity, is an *involution*. A region C which is invariant under an involution $U \neq I$ will be said to have *involutory symmetry*. Symmetry in the origin θ is one type of involutory symmetry; for this symmetry $U = -I$. The only invariant point of $U = -I$ is the origin θ. If L is any linear subspace of E_n, there exist linear involutions U which have L as a pointwise invariant subspace. For if the dimension of L is m, suppose that $\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n$ are a basis for E_n, where $\alpha_1, \ldots, \alpha_m$ are chosen to span the subspace L. Then we may define U by the equation $Ux = x_1\alpha_1 + \ldots + x_m\alpha_m - x_{m+1}\alpha_{m+1} - \ldots - x_n\alpha_n$, where $(x_1, \ldots, x_n)$ are the coordinates of any point x in E_n with respect to basis $\{\alpha_j\}$. (We may denote the matrix corresponding to any linear transformation U by the same letter U.) For the involution just defined with respect to basis $\{\alpha_j\}$, the matrix U has diagonal form with I_m, $-I_{n-m}$ on the diagonal, where I_k denotes the k dimensional identity matrix.

Conversely, we have the following:

Lemma 3.2. *Any linear involution U has a pointwise invariant subspace L, and there exists a choice of basis such that matrix U has diagonal form with I_m, $-I_{n-m}$ on the diagonal, where m is the dimension of L.*

Proof: The only possible eigenvalues of an involution are ± 1. For if $Ux = tx$, we have $U^2x = x = tUx = t^2x$, $t^2 = 1$, $t = \pm 1$. From this if follows that the basis may be chosen so that matrix U has *super-diagonal* form (elements *below* the diagonal are all zeros) with I_m, $-I_{n-m}$ on the diagonal, where m is the multiplicity of eigenvalue $+1$. (See [2].) For example, the matrix $U = \begin{vmatrix} 1 & a \\ 0 & -1 \end{vmatrix}$ has the property $U^2 = I$, and so defines an involution.

To show that the basis may always be chosen so that elements above the diagonal, as well as below, are zeros, we introduce the notion of the two associated *projections* of any involution U. A *projection* is any linear transformation P such that $P^2 = P$; since $U^2 = I$, the transformations $P = \frac{1}{2}(U \pm I)$ are projections. A pair of subspaces L_1, L_2 of E_n are *complementary subspaces* if each x in E_n has a unique expression in the form $x = x_1 + x_2$, x_1 in L_1, x_2 in L_2; uniqueness implies that $L_1 \cap L_2 = \theta$. For any projection P, the subspace $L_1 = \{x | Px = x\}$ and the null subspace $L_2 = \{y | Py = \theta\}$ are complementary subspaces, since each x in E_n has the unique expression $x = Px + (I - P)x$, Px in L_1, $(I - P)x$ in L_2. For $P = \frac{1}{2}(U + I)$, subspace L_1 is the pointwise invariant subspace of involution $U = 2P - I$, and subspace L_2 has the property that $Ux_2 = -x_2$ for all x_2 in L_2. (Accordingly we say that U is an involution *in* subspace L, *through* subspace L_2.) For any involution U, basis $\{\alpha_j\}$ may always be chosen so that $\alpha_1, \ldots, \alpha_m$ span L_1, and $\alpha_{m+1}, \ldots, \alpha_n$ span L_2; for this choice, matrix U has the required form. (If an element u_{ij} above the diagonal were not zero, the product of U into the column vector (δ_{jk}), where δ_{jk} is the Kronecker delta with j fixed, would be a column vector with u_{ij} for i'th coordinate, contrary to $U(\delta_{jk}) = \pm(\delta_{jk})$.)

Suppose that E_n is decomposed in any way into k complementary linear subspaces, $E_n = L_1 \oplus L_2 \oplus \ldots \oplus L_k$. That is, each x in E_n has a unique decomposition $x = x_1 + \ldots + x_k$, x_j in L_j, for $j = 1, \ldots, k$. We shall write also $x = (x_1, \ldots, x_k)$.

Theorem 3.3. *Let $p(x)$ be the gauge-function for a convex region C in E_n. If C has involutory symmetry through each of the complementary subspaces $L_1, \ldots, L_k$; that is, if for all $x = (x_1, \ldots, x_k)$ in E_n,*

$$p(-x_1, x_2, \ldots, x_k) = p(x_1, x_2, \ldots, x_k)$$

$$p(x_1, -x_2, \ldots, x_k) = p(x_1, x_2, \ldots, x_k)$$

$$p(x_1, x_2, \ldots, -x_k) = p(x_1, x_2, \ldots, x_k),$$

then C is symmetrical in the origin θ, *and* $p(x)$ *is a norm or pseudo-norm for* E_n.

Proof: For each $y \in E_n$, $y = (y_1, \ldots, y_k)$, apply in succession the equations in the statement of the theorem, as follows:

$p(y_1,\ldots,y_k)=p(-y_1,y_2,\ldots,y_k)=p(-y_1,-y_2,y_3,\ldots,y_k)=\ldots=p(-y_1,-y_2,\ldots,-y_k)$.

Thus for all $y \in E_n$, $p(y) = p(-y)$, which implies that p is a norm in case $p(y) > 0$ for all $y \neq \theta$; otherwise p is a pseudo-norm. (Remark: In the latter case, C is a cylinder, and is unbounded; that is, C contains entire lines of E_n. If M is the linear subspace of all x for which $p(x) = 0$, p determines a norm for the quotient-space E_n/M, and C is the Cartesian product of M and the unit sphere for the norm in E_n/M.)

Definition 3.4. If $p(x) = p(x_1, \ldots, x_n)$ is the gauge-function for a convex region $C \subset E_n$, the convex regions $C_{e_1,\ldots,e_k}$, $e_j = \pm 1$, $j = 1, \ldots, k$, corresponding to the gauge-functions

$$p_{e_1,\ldots,e_k}(x) = p(e_1x_1, \ldots, e_kx_k),$$

are called *isomers* of C, or convex regions *isomeric* to C.

Let $C_1, \ldots, C_k$ denote the sections of a convex region C in the complementary subspaces $L_1, \ldots, L_k$. Then the gauge-function for the section C_j is $p_j(x_j) = p(\theta, \ldots, \theta, x_j, \theta, \ldots, \theta)$. Conversely, if $C_1, \ldots, C_k$ are arbitrary convex regions in $L_1, \ldots, L_k$, each containing θ, with gauge-functions $p_1(x_1), \ldots, p_k(x_k)$, then there are many different convex regions C having $C_1, \ldots, C_k$ as sections. One such region is the Cartesian product of $C_1, \ldots, C_k$; another is the convex hull of $C_1, \ldots, C_k$. Let the gauge-functions be cylindrically extended to E_n: $p_j(x_j) = p_j(x) = p_j(x_1, \ldots, x_j, \ldots, x_k)$. Then evidently the gauge-function for the Cartesian product is $o(x) = \max_j p_j(x)$, and the gauge-function for the convex hull is $q(x) = \sum_{j=1}^{k} p_j(x)$.

A class of intermediate regions between the convex hull and the Cartesian product correspond to the gauge-functions

$$p(x) = [\sum_{j=1}^{k} |p_j(x)|^r]^{1/r}, \quad 1 < r < \infty.$$

(We use a similar terminology for the pointwise invariant subspace and the null subspace of a projection P, to that used for the associated involution $U = 2P - I$: P is a projection *through* the latter subspace, *onto* the former subspace.)

For a convex region C in E_n, let $D_1, \ldots, D_k$ be respectively the projections of C onto L_j through $L_1 \oplus \ldots \oplus L_{j-1} \oplus L_{j+1} \oplus \ldots \oplus L_k$, $j = 1, \ldots, k$. Then in general the projection D_j is a convex region in L_j which includes the section C_j as a proper sub-region; for the convex hull and Cartesian product of the sections, and the intermediate regions mentioned above, however, $C_1 = D_1, \ldots, C_k = D_k$.

Definition 3.5. A region C in L is of *standard type* in $L_1, \ldots, L_k$ if for each $j = 1, \ldots, k$, the projection of C through $L_1 \oplus \ldots L_{j-1} \oplus {}_{j+1} \ldots \oplus L_k$ onto L_j coincides with the section of C in L_j.

For each x in E_n, let $x = x_j + y_j$, where $x_j \epsilon L_j$, and $y_j \epsilon (L_1 \oplus \ldots L_{j-1} \oplus L_{j+1} \oplus \ldots \oplus L_k)$. Then evidently the gauge-function for the projection D_j of C is $n_j(x_j) = \inf p(x_1, \ldots, x_k)$, the infimum being taken, for fixed x_j, over all $y_j = x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k$) such that $(x_j + y_j) \epsilon C$.

Theorem 3.6. *If $C_1, \ldots, C_k$ are convex regions respectively in $L_1, \ldots, L_k$, each containing the origin θ, then any region C which has $C_1, \ldots, C_k$ respectively for its $L_1, \ldots, L_k$ sections, has a gauge-function which coincides with the gauge-functions $p_1(x_1), \ldots, p_k(x_k)$ in $L_1, \ldots, L_k$; conversely, any gauge-function which coincides with $p_j(x_j)$ in each L_j is the gauge-function for a convex region having $C_1, \ldots, C_k$ as sections in $L_1, \ldots, L_k$. In case C is of standard type in $L_1, \ldots, L_k$, the gauge-function satisfies the inequality $o(x) \leq p(x) \leq q(x)$, where $o(x)$ and $q(x)$ are as defined above; conversely, any gauge-function which satisfies this inequality is the gauge-function for a convex region of standard type, which has $C_1, \ldots, C_k$ for its $L_1, \ldots, L_k$ sections.*

Proof: Since a region C always contains the convex hull of its sections $C_1, \ldots, C_k$, it is obvious that C is of standard type if and only if it is contained in the Cartesian product of its sections. The theorem is evident from this fact, and from Remark 2.0. The following proof is given as an illustration of the use of gauge-functions.

The first statement, and converse statement, follow from the definition of gauge-function and remarks above preceding the statement of the theorem. For the second statement, suppose that C is of standard type. Then $n_j(x_j) = inf\ p(x) = p_j(x_j)$; therefore $o(x) = max\ p_j(x_j) \le p(x)$. By the triangle property of the gauge-function $p(x)$,

$$p(x_1, \ldots, x_k) \le \sum_{j=1}^{k} p_j(x_j) = q(x).$$

(The part $p(x) \le q(x)$ of the inequality is satisfied whether C is of standard type or not.) For the converse, if $x = x_j$, $o(x_j) = q(x_j)$ and $o(x_j) \le p(x_j) \le q(x_j)$; therefore the L_j section of C is C_j, $j = 1, \ldots, k$. Also $n_j(x_j) = inf\ p(x) = o(x_j) = p_j(x_j)$, since $p_i(\theta) = 0$, $i \neq j$, and $p(x) \le q(x)$, so the projection D_j coincides with C_j, $j = 1, \ldots, k$, and $C = \{x | p(x) \le 1\}$ is of standard type.

4. Rotationally Symmetric Regions.

By suitable choice of basis $\{\alpha_j\}$, the matrix R for any rotation of E_n may be put in block-diagonal form, with blocks

$$\begin{pmatrix} \cos \phi_1 & \sin \phi_1 \\ -\sin \phi_1 & \cos \phi_1 \end{pmatrix}, \ \ldots, \ \begin{pmatrix} \cos \phi_h & \sin \phi_h \\ -\sin \phi_h & \cos \phi_h \end{pmatrix}, \quad I_{n-2h},$$

on the diagonal, where I_{n-2h} is the $(n-2h)$ dimensional identity sub-matrix, all other elements of R being zeros. (See [2].) The subspace L_1 spanned by $\alpha_{2h+1}, \ldots, \alpha_n$ is left pointwise invariant by R; we say that R is a rotation of E_n *about* L. (In case $n = 2h$, no sub-identity matrix I_{n-2h} is present in the block-diagonal form; subspace L_1 is zero-dimensional, and consists of only the origin θ.) Any orthogonal transformation T of E_n, which has h pairs of conjugate complex eigenvalues $\pm\, exp\,(i\theta_j)$, has a canonical form which is the same except that I_{n-2h} is replaced by I_{n-2h-s}, $-I_s$, where s is the multiplicity of eigenvalue -1 of T.

For $(n-2h) \geq 0$, let V be any involution of an $(n-h)$ dimensional linear subspace E_{n-h} of E_n, *through* an h dimensional subspace L_2, and therefore *in* an $(n-2h)$ dimensional subspace L_1, of E_{n-h}. (In case $n = 2h$, L_1 is zero-dimensional, and consists of only the origin θ.) Suppose that $\alpha_{2h+1}, \ldots, \alpha_n$ span L_1, that $\alpha_1, \ldots, \alpha_h$ span L_2, and that $\beta_1, \ldots, \beta_h$ together with the preceding vectors span E_n. Denote the coordinates of a point of E_n, with respect to this basis, by $(x_1, y_1, \ldots, x_h, y_h, x_{2h+1}, \ldots, x_n)$, where $y_1, \ldots, y_h$ are the coefficients of $\beta_1, \ldots, \beta_h$. Let $V_1, \ldots, V_h$ denote the involutions of E_{n-h} *through* the one-dimensional subspaces respectively spanned by $\alpha_1, \ldots, \alpha_h$, *in* the $(n-h-1)$ dimensional subspaces spanned by the remaining vectors of the basis $\alpha_1, \ldots, \alpha_h, \alpha_{2h+1}, \ldots, \alpha_n$ for E_{n-h}.

If K is any convex or starlike region in E_{n-h}, which has involutory symmetry under $V_1, \ldots, V_h$ then a region C in E_n which is invariant under the rotations R, for all $\phi_1, \ldots, \phi_h$ in $0 \leq \phi_1 \leq \pi, \ldots, 0 \leq \phi_h \leq \pi$, is generated by rotating K about L_1. That is, if the column vector $(x_1, 0, \ldots, x_h, 0, x_{2h+1}, \ldots, x_n)$ is any point of K, then all points of the form

$$(x_1\cos\phi_1, -x_1\sin\phi_1, \ldots, x_h\cos\phi_h, -x_h\sin\phi_h, x_{2h+1}, \ldots, x_n)$$

are points of C. Involutory symmetry under V_j, $j = 1, \ldots, h$, is required so that points of K corresponding to $\phi_j = \pi$, $\phi_k = 0$ or π, $k \neq j$ agree with rotational symmetry under all rotations R.

Similarly, a region which is invariant under the class of orthogonal transformations T corresponding to the h dimensional torus $0 \leq \phi_1 \leq \pi$, $\ldots, 0 \leq \phi_h \leq \pi$, is obtained by rotating any region K in E_{n-h}, which is invariant under the involutions $(I_{n-h-s}, -I_s)$ and $V_1, \ldots, V_h$, about L_1 (where s is the multiplicity of eigenvalue -1 of T).

In terms of gauge-functions, if for the region K the gauge-function in $(n-h)$ coordinates is $g(x_1, \ldots, x_h, x_{2h+1}, \ldots, x_n)$, the requirement on g is that $g(\ldots -x_j \ldots) = g(\ldots x_j \ldots)$ for $j = 1, \ldots, h$, and in case of T, also that $g(\ldots -x_{n-s+1}, \ldots, -x_n) = g(\ldots x_{n-s+1}, \ldots, x_n)$. The gauge-function for C then is defined by

$$p(x_1, y_1, \ldots, x_h, y_h, x_{2h+1}, \ldots, x_n) = g(r_1, \ldots, r_h, x_{2h+1}, \ldots, x_n),$$

where $r_j = (x_j^2 + y_j^2)^{\frac{1}{2}}$, for $j = 1, \ldots, h$. Conversely, any region C which is invariant under the class of rotations R, or under the class of orthogonal transformations T, must be related to an $(n-h)$ dimensional region K with involutory symmetry, in the manner which has been described. This follows since the gauge-function p for C then determines a gauge-function g as above.

A convex or starlike region of rotation may have additional symmetry to the symmetry corresponding to invariance under all rotations of the canonical block-diagonal form with two-dimensional blocks. Although any particular rotation may be put in the canonical form, if fixed basis vectors $\{\alpha_j, \beta_j\}$ are retained, that canonical form does not represent the most general rotation about the $(n-2h)$ dimensional subspace L_1. The sections parallel to the $(2h)$-dimensional subspace L_2, perpendicular to L_1, for a region C, may in particular be solid $(2h)$-dimensional spheres, or cylinders which are products of lower dimensional solid spheres. Let π_j denote the plane spanned by α_j, β_j, $j = 1, \ldots, h$. For invariance under all rotations of the canonical form, with $\{\alpha_j, \beta_j\}$ fixed, all that is required is that the sections by the planes $\pi_1, \ldots, \pi_h$ be circles. In general C may be invariant under a class of rotations corresponding to block-diagonal matrices, with rotational blocks of dimensions $k_1, \ldots, k_{m-1}$, $k_j \geq 2$, $j = 1, \ldots, (m-1)$, and an identity or involutory diagonal sub-matrix J of dimension k_m, with $k_1 + \ldots + k_m$ ..

In terms of gauge-functions, if $g(r_1, \ldots, r_{m-1}, x_{n-k_m+1}, \ldots, x_n)$ is the gauge-function for any $[m + (n-k_m+1)]$ dimensional region K, such that $g(\ldots -r_j \ldots) = g(\ldots r_j \ldots)$, $j = 1, \ldots, (m-1)$, and $g(\ldots J(x_{n-k_m+1}, \ldots, x_n) = g(\ldots x_{n-k_m+1}, \ldots x_n)$, then the gauge-function p for a region C invariant under the larger class of rotations is given by

$$p(x_1, \ldots, x_n) = g(r_1, \ldots, r_{m-1}, x_{n-k_m+1}, \ldots, x_n),$$

where

$$r_j = [(x_{jo+1})^2 + \ldots + (x_{jo+kj})^2]^{\frac{1}{2}}, \; j = 1, \ldots, (m-1).$$

We have the following theorem concerning a region C which is invariant under a sub-interval of rotations.

Theorem 4.1. *If a region C in E_n is invariant under a set of orthogonal transformations $T(\phi)$, corresponding to all ϕ in an interval $0 \leq a \leq \phi \leq b < 2\pi$, $a < b$, where the matrix for $T(\phi)$ is in the canonical form with one variable block* $\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$, *then C is invariant under the set of orthogonal transformations $T(\phi)$ for all ϕ in the interval $0 \leq \phi \leq 2\pi$.*

Proof: Let $C(\phi)$ denote the image of C under $T(\phi)$. Let $\Delta\phi > 0$ be any increment less than $(b-a)$. Since by hypothesis $C(0) = C(a) = C(a+\Delta\phi)$, we have $C(\Delta\phi) = C(a+\Delta\phi) = C(a+2\Delta\phi) = C(0) = C(a+3\Delta\phi) = \ldots$. In this way, by suitable choice of $\Delta\phi$, any value in the interval $0 \leq \phi \leq 2\pi$ may be reached. From this the theorem follows.

5. Invariance Under Groups of Transformations.

A region C which is symmetric with respect to the origin θ is invariant under the cyclic group of order 2, which consists of the two transformations I, $-I$. A region which has involutory symmetry through complementary subspaces, as in section 3, is invariant under involutions $V_1, \ldots, V_k$ through $L_1, \ldots, L_k$; theorem 3.3 follows from the fact that the product of the transformations $V_1, \ldots, V_k$ is the negative identity. Similarly if C is invariant under any transformation T such that $T^m = -I$ for some integer m, or under any set of transformations whose product is $-I$, then C is symmetric with respect to θ.

Denote diagonal matrices, with elements $d_1, \ldots, d_n$ on the diagonal, by $[d_1, \ldots, d_n]$. If the gauge-function p for a three-dimensional region C is invariant under the diagonal involutions $U = [-1, 1, 1]$, $V = [1, -1, 1]$, $W = [-1, -1, 1]$, then C is invariant under the "Klein Fours Group", consisting of the transformations I, U, V, W, and is not necessarily symmetric with respect to θ (invariant under $-I$). An equilateral triangle is invariant under a cyclic group of order 3. If C is invariant under a cyclic group G of order 4, then C is symmetric with respect to θ if and only if G contains $-I$; this will be the case if G consists of I and the powers of a transformation Z such that $Z^2 = -I$. (For this necessarily all eigenvalues of Z are $\pm i$; therefore, the dimension must be even and every diagonal block in the canonical form must be $\begin{matrix} 0 & \pm 1 \\ \pm 1 & 0 \end{matrix}$.) If C is invariant under a group G containing a transformation W such that $W^3 = -I$ (like the cyclic group of order 6 consisting of the powers of a planar 60° rotation), then C is symmetric with respect to θ.

Theorem 4.1, together with Theorem 6.2 below, implies that the group of symmetries of any bounded convex or starlike region C, is either a discrete group (possibly consisting of only the identity), or the product of a discrete group and lower dimensional full orthogonal groups.

6. Invariance Under A Non-singular Linear Transformation

If T is any non-singular linear transformation such that the real eigenvectors of T, and the real and imaginary components of the complex eigenvectors, together span the real linear space E_n, then as shown in [2], T has a diagonal block form $[t_1A_1, \ldots, t_hA_h, t_{2h+1}, \ldots, t_n]$, where $\pm t_j \exp(i\phi_j)$, $t_j > 0$, $j = 1, \ldots, h$, t_{2h+i}, $i = 1, \ldots, (n-2h)$ are the eigenvalues of T, and $A_j = \begin{pmatrix} \cos\phi_j & \sin\phi_j \\ -\sin\phi_j & \cos\phi_j \end{pmatrix}$. If C is a convex or starlike region, with gauge-function $p(x)$, then if C is invariant under T, we must have $p(T^m x) = p(x)$ for all integers

m. The matrix for T^m is $[t_1^m A_1^m, \ldots, t_h^m A_h^m, t_{2h+1}^m, \ldots, t_n^m]$, where in case m is negative, $A_j^{-1} = A(-\phi_j) = \begin{pmatrix} \cos\phi_j & -\sin\phi_j \\ \sin\phi_j & \cos\phi_j \end{pmatrix}$. For coordinate x_{2h+1} of x, by positive homogeneity we must have, in cases $t_{2h+1} > 0$, all m, or $t_{2h+1} < 0$, m even,

$$p(0, \ldots 0, x_{2h+1}, 0, \ldots 0) = t_{2h+1}^m\, p(0, \ldots 0, x_{2h+1}, 0, \ldots 0),$$

and in case $t_{2h+1} < 0$, m odd,

$$p(0, \ldots 0, x_{2h+1}, 0, \ldots 0) = |t_{2h+1}^m|\, p(0, \ldots 0, -x_{2h+1}, 0, \ldots 0).$$

If $|t_{2h+1}| \neq 1$, this implies $p(0, \ldots 0, x_{2h+1}, 0, \ldots 0) = 0$ or ∞, so either C contains the ray $\{c\alpha_{2h+1}\}$, $c > 0$ (C contains the entire line if $t_{2h+1} < 0$), or θ is a boundary point of C for the direction α_{2h+1} (for the directions $\pm\alpha_{2h+1}$ if $t_{2h+1} < 0$). For a bounded region C with θ interior to C, we must have $|t_{2h+1}| = 1$. The same discussion applies to any coordinate x_{2h+i}, $i = 1, \ldots, (n-2h)$.

For coordinates x_1, y_1 of $x = (x_1, y_1, \ldots, x_h, y_h, x_{2h+1}, \ldots, x_n)$, by invariance and positive homogeneity we must have $p(x_1, y_1, 0, \ldots 0)$ $= t_1^m p[x_1\cos(m\phi_j) - y_1\sin(m\phi_j), x_1\sin(m\phi_j) + y_1\cos(m\phi_j), 0, \ldots 0]$, and therefore if $t_1 \neq 1$, since by Theorem 2.2 p is bounded and bounded from zero on the Euclidean unit sphere, we have $p(x_1, y_1, 0, \ldots 0)$ $= 0$ or ∞, for all x_1, y_1, not both zero. Therefore either C contains the entire plane π_1 spanned by α_1, β_1, or θ is in the boundary of C, and π_1 is exterior to or partly contains the boundary of C.

If T is an arbitrary non-singular linear transformation, then as shown in [2], T has a diagonal block form $[t_1A_1, \ldots, t_hA_h, t_{h+1}B_{h+1}, \ldots, t_kB_k]$ where $\pm t_j\, exp\,(i\phi_j)$, $t_j > 0$, $j = 1, \ldots, h$, $t_{h+1}, \ldots, t_k$ are the distinct eigenvalues of T. Each block A_j has blocks $\begin{pmatrix} \cos\phi_j & \sin\phi_j \\ -\sin\phi_j & \cos\phi_j \end{pmatrix}$ on its diagonal, all elements below the diagonal blocks being zeros. Each block B_{h+i} is superdiagonal, with all 1's on the diagonal. If C is invariant under T, it is also invariant under T^{-1}, and the eigenvalues of T^{-1} are reciprocals of the eigenvalues of T. Therefore in the discussion of any particular eigenvalue, we may suppose that $t_j > 1$ or $|t_{h+i}| > 1$, if they are not equal to 1. In case say α_1, β_1, ..., α_3, β_3, ... span the eigensubspace of a single pair of conjugate complex eigenvalues $\pm t_1 \, exp \, (i\phi_1)$, we have

$$p(0, \ldots 0, x_3, y_3, 0, \ldots 0) = t_1^m \, p[x_1(m)/t_1, y_1(m)/t_1, \ldots, x_1\cos(m\phi_1) - y_1\sin(m\phi_1), x_1\sin(m\phi_1) + y_1\cos(m\phi_1), 0, \ldots 0],$$

and since if C is bounded, with θ interior to C, the function p is bounded from zero on the Euclidean unit sphere, the expression $p[\ ]$ on the right is bounded from zero, and $t_1^m \, p[\ ]$ approaches infinity as increases. But $p(m)$ on the left is fixed, so we have a contradiction, and if $t_1 > 1$, either C must contain the entire eigensubspace of $\pm t_1 \, exp \, (i\phi_1)$ (in case $p = 0$), or the eigensubspace must be exterior to the interior of C (in case $p = \infty$). Similarly for the real eigenvalues, if C is bounded, with θ interior to C, we must have $|t_{h+i}| = 1$. We have now established the following theorem.

Theorem 6.1. *If a starlike or convex region C, with θ interior to C, is invariant under a linear transformation T, then the eigenvalues of T are all of modulus 1.*

Following is a further theorem on the nature of a linear transformation T under which a convex or starlike region C may be invariant.

Theorem 6.2. *If a starlike or convex region C, with θ interior to C, is invariant under a linear transformation T, then T must be such that the eigenvectors of T span E_n. By Theorem 6.1. this implies that for any T under which a C can possibly be invariant, there always exists a choice of basis for E_n with respect to which T appears as an orthogonal transformation.*

Proof: Consider the powers $(B_{h+1})^m$ of a block in the canonical form for T. If B_{h+1} is of dimension s, it may be easily verified that the elements respectively in row 1, column 2, and in row $(s-1)$, column s of $(B_{h+1})^m$ are $mb_{12}, mb_{s-1,s}$, where b_{12}, $b_{s-1,s}$ are the corresponding elements of B_{h+1}. By a similar argument to those used to prove Theorem 6.1, it follows from invariance under T^m, and from boundedness and boundedness from zero of p, that $b_{12} = b_{s-1,s} = 0$. Likewise the other elements of B_{h+1} above the diagonal must be zeros, and the elements above the diagonal blocks in A_1 must be zeros. The same discussion applies to the other blocks A_j, B_{h+1}; therefore the basis vectors corresponding to the real eigenvalues are all eigenvectors, and the basis vectors $\{\alpha_j, \beta_j\}$ corresponding to the complex eigenvalues are all components of complex eigenvectors, which is the theorem.

Finally, we have the following theorem relating transformation by a non-involutory orthogonal transformation, and transformation by an involution.

Theorem 6.3. *For any region C, if T is any orthogonal transformation which is not an involution, then there exists an isomer J(C) of C, such that TC equals UJ(C), where U is an involution.*

Proof: Let involution J be the involution having matrix $[1, -1, \ldots, 1, -1, 1, \ldots, 1]$, where there is one pair $(1,-1)$ corresponding to each block $\begin{pmatrix} \cos \phi_j & \sin \phi_j \\ -\sin \phi_j & \cos \phi_j \end{pmatrix}$ in the canonical matrix for T. Then $U = TJ$ is the required involution, since $J^2 = I$, and since the product of the rotational block by the column matrix $(1,-1)$ is the block $\begin{pmatrix} \cos \phi_j & -\sin \phi_j \\ -\sin \phi_j & -\cos \phi_j \end{pmatrix}$, which is involutory. This may be seen also from the fact that J has the property $J^{-1}TJ = JTJ =$ [illegible], or $TJ = JT^{-1}$: $U^2 = TJJT^{-1} = I$. For the same J and U, we have also $TJ(C) = U(C)$. Likewise $V = JT = T^{-1}J$ is an involution, since $V^2 = JTT^{-1}J = I$, so we have $JT(C) = V(C)$. (The isomer of the transform of C by T is the same as an involutory transform of C.)

7. Composite Regions

The process, described in section 4, of generating an n dimensional region by rotation of an $(h+s)$-dimensional region, in subspaces of dimensions $h_1, \ldots, h_s$, $h + h_1 + \ldots + h_s = n$, may be generalized as indicated in the following definition.

Definition 7.1. An n dimensional region C is *composite* if its gauge-function p may be expressed in terms of lower dimensional gauge-functions, as follows:

$$p(x_1, \ldots, x_n) = p(q_1, \ldots, q_s, x_{n-h+1}, \ldots, x_n),$$

where

$$q_1 = q_1(x_1, \ldots, x_{h_1}), \ldots, q_s = q_s(x_{n-h+1-h_s}, \ldots, x_{n-h+1-1}),$$

are gauge-functions of respective dimensions $h_1, \ldots, h_s$.

In the special case of rotationally symmetric regions, the gauge-functions $q_1, \ldots, q_s$ are gauge-functions of spheres, with origin θ at the centers of the spheres.

Definition 7.2. An n dimensional region C is *step-composite* if, for some partition of the h_1-, $\ldots$, h_s-dimensional solid spheres into h_1-, $\ldots$, h_s-dimensional "solid angular" portions, the gauge-function of C may be represented in the above form in each of the portions, different gauge-functions $q_1, \ldots, q_s$ being allowed in the several portions. A region C is *partially composite* if its gauge-function may be represented in the above form in one or several "solid angular" portions, not comprising all of the set of h_1-, ..., h_s-dimensional solid spheres.

Following are examples of the above types of regions:

Examples 7.3. Let $q(x_1,x_2)$ be the gauge-function for any two-dimensional convex region having the origin θ in its interior, and let $p(z,x_3)$ be the two-dimensional gauge-function $max\ (|z|,|x_3|)$. Then $p(x_1,x_2,x_3) = p[q(x_1,x_2),\ x_3]$ is the gauge-function for a composite three-dimensional region. The region is a cylinder with sections parallel to the (x_1,x_2)-plane congruent to the region $q(x_1,x_2) \leq 1$. If $p(x,x_3)$ is $|x| + |x_3|$ instead of $max\ (|z|,|x_3|)$, the composite three-dimensional region is the convex hull of the set $q(x_1,x_2) \leq 1$ and the points (0,0,1), (0,0,-1). In this case the sections parallel to the (x_1,x_2)-plane are similar to the region $q(x_1,x_2) \leq 1$.

Examples 7.4. In E_3, the points $P_1 = (1,0,-2^{\frac{1}{2}}/4)$, $P_2 = (-\frac{1}{2}, 3^{\frac{1}{2}}/2, -2^{\frac{1}{2}}/4)$, $P_3 = (-\frac{1}{2}, -3^{\frac{1}{2}}/2, -2^{\frac{1}{2}}/4)$, $P_4 = (0,0,3.2^{\frac{1}{2}}/4)$ are vertices of a regular tetrahedron of edge $3^{\frac{1}{2}}$. Let $g_1(x)$, $g_2(x)$, $g_3(x)$, $g_4(x)$ be respectively the gauge-functions for spheres of radius $3^{\frac{1}{2}}$ with centers at P_1, P_2, P_3, P_4. Then $r(x) = max[g_1(x),\ g_2(x),\ g_3(x),\ g_4(x)]$ is a convex function; the region $D = \{x | r(x) \leq 1\}$ is the common portion of the four spheres, and clearly is step-composite. For an example of a partially composite region, we may take the convex hull of the intersection of D with the half-space $x_3 \leq 0$, and the point (0,0,1).

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THE COEFFICIENTS OF $\sinh x/\sin x$

By L. Carlitz

Put

(1) $$x/\sin x = \sum_{m=0}^{\infty} (-1)^m D_{2m} x^{2m}/(2m)!\,,$$

where

(2) $$D_{2m} = 2(1 - 2^{2m-1})B_{2m};$$

the notation is that of Norlund [3, Ch. 2].

By the Staudt-Clausen theorem for the Bernoulli numbers

(3) $$B_{2m} = G_{2m} - \tfrac{1}{2} - \sum_{p-1|2m} 1/p \qquad (m \geq 1),$$

where G_{2m} is an integer and the summation is extended over odd primes p such that $p - 1|2m$. Using (2) and (3), we get

(4) $$D_{2m} = H_{2m} - \sum_{p-1|2m} 1/p \qquad (m \geq 1)$$

where again H_{2m} is an integer and the summation is over odd primes p.

If we now write

(5) $$\frac{\sinh x}{\sin x} = \sum_{m=0}^{\infty} \beta_{2m} \frac{x^{2m}}{(2m)!}$$

then it follows from (1) that

(6) $$\beta_{2m} = \frac{1}{2m+1} \sum_{s=0}^{m} (-1)^s \binom{2m+1}{2s} D_{2s}.$$

In particular $\beta_2 = 2/3$, $\beta_4 = 4/3$, $\beta_6 = -148/21$, $\beta_8 = -17744/9$.

Incidentally (5) implies

$$\frac{\sin x}{\sinh x} = \sum_{m=0}^{\infty} (-1)^m \beta_{2m} \frac{x^{2m}}{(2m)!}$$

so that

$$\sum_{s=0}^{m} (-1)^s \binom{2m}{2s} \beta_{2s}\beta_{2m-2s} = 0 \qquad (m \geq 1).$$

Also using the well known formula

$$B_{2m} = \sum_{k=0}^{2m} \frac{1}{k+1} \sum_{s=0}^{k} (-1)^s \binom{k}{s} s^{2m} ,$$

a little manipulation leads to the explicit formula

$$(2m+1)\beta_{2m} = \sum_{k=0}^{2m} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{s=0}^{m} (-1)^s \binom{2m+1}{2s} (2-2^{2s}) j^{2s}$$

$$= \sum_{k=0}^{2m} \frac{1}{k+1} \sum_{s=0}^{k} (-1)^s \binom{k}{s} \{(1+is)^{2m+1} + (1-is)^{2m+1}$$

$$- \tfrac{1}{2}(1+2is)^{2m+1} - \tfrac{1}{2}(1-2is)^{2m+1}\} .$$

It is natural to ask whether a result like (4) holds for β_{2m}; as we shall see this is not the case.

Let

$$(8) \qquad p^r | 2m+1, \quad p^{r+1} | 2m+1.$$

It is familiar that if $p^k | s$, $p-1 | 2s$ then

$$(9) \qquad B_{2s} = 0 \qquad (\text{mod } p^k);$$

hence D_{2s} also satisfies this congruence. A term in the right member of (6)

$$\binom{2m+1}{2s} D_{2s} = \frac{2m+1}{2s} \binom{2m}{2s-1} D_{2s} ,$$

with $p-1 \nmid 2s$, is therefore divisible by p^r. Thus (6) reduces to

$$(10) \qquad (2m+1)\beta_{2m} \equiv 1 + \sum_{\substack{s=1 \\ p-1|2s}}^{m} (-1)^s \binom{2m+1}{2s} D_{2s}$$

$$\equiv 1 + \sum_{\substack{s=1 \\ p-1|2s}}^{m} (-1)^s \binom{2m+1}{2s} \left(D_{2s} + 1/p - 1 \right)$$

$$- \sum_{\substack{s=1 \\ p-1|2s}}^{m} (-1)^s \binom{2m+1}{2s} \left(1/p - 1 \right) \quad (\text{mod } p^r).$$

Analogous to (9) we have also [1, Th. 3]

(11) $$B_{2s} + 1/p - 1 \equiv 0 \pmod{p^k} \qquad (p^k(p-1)|2s).$$

Using (2) we get

(12) $$B_{2s} + 1/p - 1 \equiv 0 \pmod{p^k} \quad (p^k(p-1)|2s).$$

Hence exactly as above

$$\binom{2m+1}{2s}\left(D_{2s} + 1/p - 1\right) \equiv 0 \pmod{p^r}$$

for $p - 1|2s$, $s \geq 1$. Consequently (10) becomes

(13) $$(2m+1)\beta_{2m} \equiv 1 - \sum_{\substack{s=1 \\ p-1|2s}}^{m} (-1)^s \binom{2m+1}{2s}\left(1/p - 1\right) \pmod{p^r}.$$

We now consider separately the cases $p \equiv 1$, $p \equiv 3 \pmod 4$. In the first case it is evident that s is even so that (13) may be written

(14) $$(2m+1)\beta_{2m} \equiv 1 - \sum_{0<s(p-1)<2m+1} \binom{2m+1}{s(p-1)}\left(1/p - 1\right) \pmod{p^r}.$$

But it has been proved [1] that if $p^r|n$ then

(15) $$p + (p-1) \sum_{0<s(p-1)<n} s(p^n - 1) \equiv 0 \pmod{p^{r+1}}.$$

Clearly (14) and (15) imply

(16) $$(2m+1)\beta_{2m} \equiv 0 \pmod{p^r},$$

in other words β_{2m} is integral (mod p),

On the other hand for $p \equiv 3 \pmod 4$, we have

(17) $$(2m+1)\beta_{2m} \equiv 1 - \sum_{0<s(p-1)<2m+1} (-1)^s \binom{2m+1}{s(p-1)}\left(1/p - 1\right) \pmod{p^r}$$

For example when $2m + 1 = p$, (17) becomes

(18) $$p\beta_{p-1} \equiv 2 \pmod{p};$$

indeed for $2m + 1 = p^r$ we find that

(19) $$p^r \beta_{p^r-1} \equiv 2 \pmod{p} p$$

so that the denominator of β_{p^r-1} is divisible by exactly p^r.

According to (17), the highest power of p dividing the denominator of β_{2m} does not exceed p^{r+1}. We shall now show that this case does indeed occur. Clearly the sum

$$\sum_{0<j(p-1)<2m+1} (-1)^j \binom{2m+1}{j(p-1)} \equiv -\sum_{s=0}^{p-1} (1+is)^{2m+1} \pmod{p},$$

as is easily verified by expanding the right member and summing over s. Now let

(21) $2m+1 \equiv u + vp \pmod{p^2-1}$ $(0 \le u \le p-1;\ 0 \le v \le p-1)$.

Since $(1+is)^p \equiv 1 - is$, $(1+is)^{p^2-1} \equiv 1 \pmod{p}$, the right member of (20) becomes

(22) $$-\sum_{s=0}^{p} (1 \quad is)^u (1-is)^v$$

which on expansion yields

(23) $$-\sum_j (-1)^j \binom{u}{j} \binom{v}{p-1-j},$$

provided $u + v \ge p - 1$, otherwise the sum (22) $\equiv 0 \pmod{p}$. Now if $w = p - 1 - v$

$$\binom{p-1-w}{p-1-j} = \frac{(p-1-w)!}{(p-1-j)!\,(j-w)!} \equiv (-1)^{j+k} \frac{j!}{w!\,(j-w)!},$$

so that (23)

$$\equiv -(-1)^w \sum_j \binom{u}{j}\binom{j}{w} \equiv -(-1)^w \binom{u}{w} \sum_j \binom{u-w}{j-w}$$

$$\equiv -(-1)^w \binom{u}{w} 2^{u-w} \equiv -\frac{u!\,v!}{(u+v-p+1)!} 2^{u+v} \pmod{p}.$$

This result can also be obtained by means of (7).

To summarize we state the following:

Theorem. *The denominator of* β_{2m} *contains only primes* $p \equiv 3 \pmod 4$. *Let* $p^r | 2m + 1$, $p^{r+1} \nmid (2m + 1)$ *and define* u, v *by means of* (21). *Then* β_{2m} *satisfies* (17). *If* $u + v \geq p$ *then the denominator of* β_{2m} *is divisible by exactly* p^{r+1} *and indeed*

$$p^{r+1} \beta_{2m} \equiv \frac{u!v!}{(u + v - p + 1)!} \quad 2^{u+v} \pmod p,$$

If $u + v < p$ *the denominator of* β_{2m} *is divisible by* p^r *at most.*

The above results do not indicate when the denominator is exactly divisible by p^k, where $0 \leq k \leq r$, although (19) gives some information for $k = r$.

It may also be remarked that results similar to the above hold for the coefficients in the power series expansions of

$$\frac{(e^{wx} - 1)}{(e^x - 1)},$$

where w is a primitive n-th root of unity.

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Duke University

MISCELLANEOUS NOTES

Edited by
Charles K. Robbins

Articles intended for this Department should be sent to Charles K. Robbins Department of Mathematics Purdue University, Lafayette, Indiana.

FAREWELL TO 1955

(popular)

J. M. Gandhi

The year has just gone. A mathematician has bid good-bye to it as follows.

Mathematician –*X*. "Hello! 1955. We are sorry to see you go, we are however very glad that you have given us many remarkable theories and inventions – e.g. some of these are:

$$\begin{aligned}1955 = {} & 1^5 + 2^5 + 3^5 + 4^5 + x + x \\ & + 1^4 + 2^4 + 3^4 + 4^4 + x + x \\ & + 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + x \\ & + 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + x \\ & + 1 + 2 + 3 + 4 + 5 + 6.\end{aligned}$$

$$\begin{aligned}1955 = {} & 5^5 - 5^4 - 5^3 - 5^2 - 5^1 \\ & - 4^4 - 4^3 - 4^2 - 4^1 - 4^0 \\ & - 3^3 - 3^2 - 3^1 - 3^0 - x \\ & - 2^2 - 2^1 - 2^0 - x - x \\ & - 1^1 - 1^0 - x - x - x.\end{aligned}$$

$$\begin{aligned}1955 = {} & 1 \times 4^5 + (1+2) \times 3^5 + (1+2+3) \times 2^5 \\ & + (1+2+3+4) \times 1^5.\end{aligned}$$

$$1955 = 3^7 - 3^5 + 3^3 - 3^1 - 2^4 + 2^2 - 2^0.$$

$$\begin{aligned}1955 = {} & + 12^2 + 13^2 + 14^2 + 15^2 + 16^2 + 17^2 \\ & + 18^2 + 19^2 - 3^2.\end{aligned}$$

Magic Square of the Year 1955

1	9	5	5
9	1	5	5
1	9	5	5
9	1	5	5

Sum from any side is equal to 20.

$$\text{i.e. sum} = \frac{6}{1} + \frac{28}{2} = \frac{\text{1st perfect number}}{1} + \frac{\text{2nd perfect number}}{2}.$$

Some more properties:

$$\begin{aligned} 1955 = {} & 10^3 + 8^3 + 6^3 + 4^3 + 2^3 \\ & + 8^2 + 6^2 + 4^2 + 2^2 + 0^2 \\ & + 10^1 + 8^1 + 6^1 + 4^1 + 2^1 \\ & + 8^0 + 6^0 + 4^0 + 2^0 + 0^0 \quad (\text{if } 0^0 = 1). \end{aligned}$$

$$\begin{aligned} 1955 = {} & 9 - 10 + 11 - 12 + 13 + 10^3 - 11^3 \\ & + 12^3 - 13^3 + 14^3. \end{aligned}$$

So friend, good-bye. I have paid a very humble contribution to the progress of Mathematics and to the progress of science and peace in the world. Let us hope my successor will be still more wonderful."

WELCOME TO 1956!

It is left for the reader of this note to give a suitable welcome to the year 1956.

Lingraj College Belgaum, South India.

A GENERALIZATION OF THE S-STIRLING NUMBERS

Jerome Hines

Introduction: The purpose of this paper is to develope a general formula for the S-Stirling numbers by the use of poly-gamma functions. Since the poly-gamma functions have meaning also for non-integer values, we can take the resulting expression for the S-Stirling numbers as their definition. Then we will have defined a Stirling function which is a generalization of the Stirling numbers and will include them as a special case.

1. The T-Stirling numbers can be found by the formula

1.1 $$ {}_{n+1}T_p = \sum_{i=0}^{n} (-)^i \frac{(n-i+1)^{p+n}}{\Gamma(n-i+1)\,\Gamma(i+1)} $$

and have the recursion formula

1.2 $$ {}_{n+1}T_p = (n+1)\ {}_{n+1}T_{p-1} + {}_nT_p $$

The S-Stirling numbers have the recursion formula

1.3 $$ {}_{n+1}S_p = (n+1)\ {}_nS_{p-1} + {}_nS_p $$
$$ = 0, \quad n < p $$

and can be generated as follows:

1.4 $$ (1+x)(1+2x)\ \ldots\ (1+nx) = {}_nS_0 + {}_nS_1x + {}_nS_2x^2 + {}_nS_3x^3 + \ldots $$

The following is a table of some of the simpler S-Stirling numbers:

${}_nS_p$ (p ↓, n →)	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	0	1	3	6	10	15	21	28
2	0	0	2	11	35	85	175	322
3	0	0	0	6	50	225	735	1960
4	0	0	0	0	24	274	1624	6769
5	0	0	0	0	0	120	1784	11352

In the following section a general formula for ${}_nS_p$ will be developed.

2. From the recursion formula, 1.3,

$$ {}_nS_n = {}_{n-1}S_n + n \; {}_{n-1}S_{n-1} $$

But

$$ {}_{n-1}S_n = 0 $$

whence

$$ {}_nS_n = n \; {}_{n-1}S_{n-1} $$

Continuing:

$$ {}_nS_n = n(n-1) \; {}_{n-2}S_{n-2} $$

Further:

$$ {}_nS_n = \Gamma(n+1) \; {}_0S_0 $$

But

$$ {}_0S_0 = 1 $$

whence

2.1 $$ {}_nS_n = \Gamma(n+1) $$

Similarly

$$ \begin{aligned} {}_nS_{n-1} &= {}_{n-1}S_{n-1} + n \; {}_{n-1}S_{n-2} \\ &= \Gamma(n) + n \left[{}_{n-2}S_{n-2} + (n-1) \; {}_{n-2}S_{n-3} \right] \\ &= \Gamma(n) + n\Gamma(n-1) + n(n-1)\,\Gamma(n-2) + \dots \\ &\quad + \Gamma(n+1) \; {}_1S_0 \end{aligned} $$

Then

2.2 $$ {}_nS_{n-1} = \Gamma(n+1) \left[\frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{2} + 1 \right] $$

Let us define $\psi_{(r)}(n+1)$ by

2.3 $$ \begin{aligned} \psi_{(r)}(n+1) &= \frac{1}{\Gamma(n+1)} D_n{}^r \, \Gamma(n+1) \\ &= \frac{1}{\Gamma(n+1)} \int_0^\infty e^{-x} x^n (lg\cdot x)^r \, dx \\ &= \frac{1}{\Gamma(n+1)} \left[-e^{-x} x^n (lg\cdot x)^r \right]_{x=0}^{x=\infty} \end{aligned} $$

$$ + \frac{1}{\Gamma(n+1)} \int_0^\infty e^{-x} \left[nx^{n-1}(lg\cdot x)^r + rx^{n-1}(lg\cdot x)^{r-1} \right] dx $$

For r positive and finite the first term of this expression is zero. Then

$$\psi_{(r)}(n+1) = \frac{n}{n\Gamma(n)} \int_0^\infty e^{-x}\, x^{n-1}(lg\cdot x)^r\, dx$$

$$+ \frac{r}{n\Gamma(n)} \int_0^\infty e^{-x}\, x^{n-1}(lg\cdot x)^{r-1}\, dx$$

2.4 $$\psi_{(r)}(n+1) = \psi_{(r)}(n) + \frac{r}{n}\,\psi_{(r-1)}(n)$$

whence

$$\psi_{(r)}(n+1) = \psi_{(r)}(n-1) + \frac{r}{n-1}\,\psi_{(r-1)}(n-1)$$

$$+ \frac{r}{n}\,\psi_{(r-1)}(n)$$

Continuing this process:

2.5 $$\psi_{(r)}(n+1) = \psi_{(r)}(1) + r\sum_{j=1}^{n} \frac{\psi_{(r-1)}(j)}{j}$$

Note that $\psi_{(1)}(n+1)$ is the ordinary digamma function in which $\psi_{(1)}(1) = -C$, Euler's constant. If we define the ξ-functions by

2.6 $$\xi_{(r)}(n+1) = \psi_{(r)}(n+1) - \psi_{(r)}(1)$$

then in the case of $r = 1$,

$$\xi_{(1)}(n+1) = \psi_{(1)}(n+1) + C$$

2.7 $$\xi_{(1)}(n+1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

and

2.8 $$\xi_{(r)}(n+1) = r\sum_{j=1}^{n} \frac{\xi_{(r-1)}(j)}{j}$$

Let us now put equation 2.2 in the form

$${}_nS_{n-1} = \Gamma(n+1)\;\xi_{(1)}(n+1)$$

Continuing the process used in developing 2.2,

$${}_nS_{n-2} = \Gamma(n+1)\left[\frac{\xi_{(1)}(n)}{n} + \frac{\xi_{(1)}(n-1)}{n-1} + \frac{\xi_{(1)}(n-2)}{n-2} + \cdots\right]$$

$$= \frac{\Gamma(n+1)}{2!}\,\xi_{(2)}(n+1)$$

Generally:

$$ {}_nS_{n-r} = \Gamma(n+1)\left[\frac{\xi_{(r-1)}(n)}{n} + \frac{\xi_{(r-1)}(n-1)}{n-1} + \cdots\right] $$

or

2.9
$$ {}_nS_{n-r} = \frac{\Gamma(n+1)}{r!}\,\xi_{(r)}(n+1) $$

and

2.10
$$ {}_nS_p = \frac{\Gamma(n+1)}{\Gamma(n-p+1)}\,\xi_{(n-p)}(n+1) $$

3. The following are two examples of actual calculation of S-Stirling numbers by 2.9:

a). Determine ${}_4S_3$ using

$$ {}_nS_{n-1} = \Gamma(n+1)\ \xi_{(1)}(n+1): $$

$$ {}_4S_3 = 4!\ \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) $$
$$ = 24 + 12 + 8 + 6 $$
$$ = 50 $$

b). Determine ${}_4S_2$ using

$$ {}_nS_{n-2} = \frac{\Gamma(n+1)}{2!}\ \xi_{(2)}(n+1): $$

$$ {}_4S_2 = \frac{4!}{2!} \times 2!\left[\frac{\xi_{(1)}(2)}{2} + \frac{\xi_{(1)}(3)}{3} + \frac{\xi_{(1)}(4)}{4}\right] $$
$$ = 4!\left[\frac{1}{2} + \frac{1+\frac{1}{2}}{3} + \frac{1+\frac{1}{2}+\frac{1}{3}}{4}\right] $$
$$ = 12 + 8 + 4 + 6 + 3 + 2 $$
$$ = 35 $$

4. The main value of the expression

4.1
$$ {}_nS_p = \frac{\Gamma(n+1)}{\Gamma(n-p+1)}\ \xi_{(n-p)}(n+1) $$

is that it relates the S-Stirling numbers to the polygamma functions and also gives a method by which we can define a continuous S-Stirling function for n and p not integers through the definitions of the polygamma functions; i.e. 4.1 can be taken as the definition of the S-Stirling function of two variables, n and p.

CURRENT PAPERS AND BOOKS

Edited by

H.V. Craig

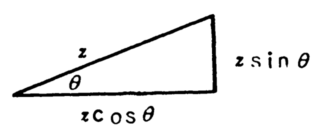
This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H.V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Comment on H.W. Becker's "Comment on L. Caners' "'Pythagorean Principle and Calculus'"*

According to H.W. Becker "the petitio Principii" is that in taking $d_\theta\ (uz^{-2}) = 0$, the author has assumed uz^{-2} constant, in advance of proving it. In the present paper I am going to prove it in detail.



Consider the right angled triangle with θ and z variables.

Let

$$z^2 \cos\theta + z^2 \sin\theta = u \tag{1}$$

Then

$$\cos^2\theta + \sin^2\theta = uz^{-2}\ .$$

Differentiating both sides with respect to θ we get.

$$2\cos\theta \frac{d}{d\theta}(\cos\theta) + 2\sin\theta\frac{d}{d\theta}(\sin\theta) = d_\theta\ (uz^{-2}) \tag{2}$$

"The Petitio Principii" is that Leonard Caners has assumed

$$\left.\begin{aligned} \frac{d}{d\theta}(\cos\theta) &= -\sin\theta \\ \frac{d}{d\theta}(\sin\theta) &= \cos\theta \end{aligned}\right\} \tag{3}$$

in advance of proving these results without the use of the Phythagorean Principle.

Now from the definition of differentiation, we have

* MATHEMATICS MAGAZINE, Vol. 28 (1955) p. 276 and Vol. 29 p. 40.

$$\frac{d}{d\theta}\sin\theta = \lim_{h\to 0}\frac{\sin(\theta+h)-\sin\theta}{h} \tag{4}$$

If

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \tag{5}$$

and

$$\lim_{h\to 0}\frac{\sin h/2}{h/2} = 1 \tag{6}$$

From (4) we have

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta$$

The remaining problem is to prove (5) without using the Pythagorean Principle.

Since Trigonometry is based on right angled triangles, at first we may think that (5) can not be derived without using the Pythagorean Principle.

But if we see the proof of (5) in any elementary book on trigonometry it will be evident that no pythagorean principle is used in deriving it. Only the definitions of sin θ and cos θ are used.

Exactly similar discussions are needed in proving

$$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

We now use

$$\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$$

This can also be proved without use of the Pythagorean Principle.

Making these substitutions in (3) we have

$$-2\cos\theta\sin\theta + 2\cos\theta\sin\theta = D_\theta(uz^{-2})$$

$$\therefore 0 = D_\theta(uz^{-2})$$

$$uz^{-2} = \text{Cons.}$$

J. M. Gandhi - Belgaum, India.

Comment On Pedro Pizo's "On The Case n = 3 of Fermat's Last Theorem"

In glancing through the January-February issue of 1955, I stopped to read "On the Case $n = 3$ of Fermat's Last Theorem" by Pedro A. Piza. although it is rather late to make comments on it, I must say that I fail to follow the reasoning. In fact, I believe that Mr. Piza has made an error. He begins by showing that the cube of a number of the form $a^2 + 3b^2$ is of the same form. This is all right. However, when he attempts to use this fact in his proof he says, "If $p^2 + 3q^2$ is a cube it is a cube of an integer of the same type..." In other words, he uses the converse of the lemma rather than the lemma itself. But he does not prove the converse, as far as I can see. If I am mistaken, I should like to be informed ofmmy error.

Peter Yff

Carl Friedrich Gauss: Titan of Science. By Dr. G. Waldo Dunnington, Exposition Press, New York, December 1955. Illustrated - $6.00.

Dr. G. Waldo Dunnington, a member of the faculty of Northwestern State College, is the authod of a new biography of Friedrich Gauss.

On February 23, 1855, Carl Friedrich Gauss, one of the three greatest mathematical geniuses of all time, died in Göttingen, Germany, where he had been director of the Observatory of the University most of his life. The centennial year 1955, which brought several memorial celebrations at Göttingen and elsewhere in Germany, is now climaxed by the publication of the first full-scale biography of Gauss, written by the foremost American Gauss scholar, a participant in the various observances in Europe.

Harold Weiss

Comment on Mr. Diamond's "Irrational Numbers"

In his article "Irrational Numbers" (Mathematics Magazine Vol. 29 No. 2, page 89) Mr. Diamond mentions on Page 97 Felix Klein's derivation of the word "irrational." I would like to comment on it, because philosophy of Mathematics is involved in this derivation. Furthermore the question arises whether the word irrational is a misnomer in our own language.

It is correct that the Latin word "irrationalis" is the translation of the Greek word "alogos." But this word means something very much deeper than "inexpressible." According to ancient legends the expression "alogos nomos" for "Irrational numbers" was introduced by Pythagoras who nad found that the diagonal of a square is incommensurable to its side. The concept of incommensurability in the realm of geometry is equivalent to the concept of irrationality in the realm of algebra. Pythagoras visualized numbers in the form of geometrical patterns.

Here he faced an astonishing fact. Nothing seemed simpler than to draw a diagonal in a square with the side of the unit length. The points to be connected seemed well defined. But the lenth of the diagonal defied any clear cut measurement in regard to unity.

The human mind could not - according to Pythagoras' philosophy of numbers - conceive the new unit "square root of 2." Only what the human mind could conceive was in the domain of Man, was part of the logos. Logos is the highest expression of the ability of constructive thinking. The derivation "logical" in our language expresses its meaning better than any definition. So. St. John explains the origin of all things by writing: "In the beginning was the logos." When we find in the Bible the translation: "In the beginning was the Word," we should not forget that "Word" in this connection means the expression of God by which He may communicate with Man. It means revelation to the mind, not spoken symbols of thought by human beings.

So "alogos," the contradiction to logos, does not mean "inexpressible" but "inaccessible to the reasoning mind". Since it belonged to the domain of the Gods, Pythagoras forbade the students and disciples of his school to talk about irrational numbers in public.

Many centuries later Euclid "dethroned" irrational numbers from their lofty height. From then on they belonged to the domain of Man. But it was too late to change the word alogos. It became a very "logical" mathematical expression and then and there a misnomer in this field.

But Language itself was not affected. The Latin translation of logos is "ratio." From this word our language inherited "reason" and formed "rational" and "irrational," meaning defined or not defined by reason.

Also the Latin word "ratio" was "liberated" from its Latin origin, meaning, and pronounciation and became the English household word as used in Mathematics. From this word "ratio" the adjectives "rational" and "irrational" are derived.

These words, as psychological and mathematical expressions, are simply cousins with the common ancestor, the Latin word "ratio." A misnomer happened almost 2000 years ago. Today "rational" and "irrational" are expressions for different concepts in two different fields of science.

Finally, to complete the picture, we have a similar development of two other words in more recent times. Imaginary numbers once were not considered numbers. So we have to this day the distinction between real and imaginary numbers, knowing that imaginary numbers are not "imaginary" at all, but as "real" as real numbers belonging to one complete number system.

Fred G. Elston

Elementary Topology. By Dick Wick Hall and Guilford L. Spencer II. John Wiley & Sons, New York. 303 pp. $7.00

An introduction to general point-set topology. the new text includes a variety of topics previously slighted or overlooked. Hall and Spencer accord full coverage to the Jordan Curve Theorem and provide considerable material on metric spaces and non-metric or general spaces. In addition, they include a characterization of the sphere, which intrinsically contains a characterization of the plane.

At the outset, the authors put the student to grips with infinite, countable and uncountable sets, and the real number system, rather than taking these topics for granted. Similarly, the Axiom of Choice is treated in detail and an introduction provided to Bing's results in the theory of Partionable Spaces.

Chapter headings include: introductory set theory, the real line, topological spaces, metric spaces, arcs and curves, partionable spaces, and the axiom of choice. Over 400 problems are graded from elementary to those suitable for short term papers.

Richard Cook

AN ERROR CORRECTED

A typographical error was made in '*A Grass Root Origin of a Certain Mathematical Concept*,' Jan.-Feb. 1956, page 132, second paragraph fourth line. It should read:

surfaces run $3/2$, $(3/2)^2$, $(3/2)^3$ ... of the original surface. Thus....

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher,* are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Aflred, New York.

NEW MEANINGS FOR OLD SYMBOLS

(popular)

Louis E. Diamond

In our early school days mathematics simply means the manipulation of number symbols in arithmetic, and the manipulation of points, lines, and shapes in geometry. At that time we do not relate these two subjects to each other except in so far as our teachers classify them as subjects under the vague heading "Mathematics." We are so accustomed to consider the symbols 1, 2, 3, . . . and so on as numbers that we are apt to forget or perhaps we never realized, that these symbols need not necessarily represent numbers, nor need the symbols + and × necessarily denote the ordinary arithmetical operations of addition and multiplication.

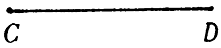
As a simple example, when we write $2+3=5$, we are so accustomed to the operations of arithmetic that we entirely forget that 2, 3, and 5 are merely symbols invented by man, and that + indicates a binary operation that is, an operation by which two of the symbols are combined according to certain rules. We also forget that the symbol = is only a symbol of relation between the symbols on the right hand side and the symbols on the left hand side. In our mind we feel that there is something about this statement that $2+3=5$ which is so fundamental that even questioning it is absurd. It is a statement that is true and that is that. The abstract symbols have become concrete objects per se and represent to us the only possible kind of addition. We do not question the meaning of the symbols used. They are sacrosanct.

Now the fact is that these symbols may also represent operations of various kinds. The symbols + and × may likewise stand for operations entirely different from those we are accustomed to in arithmetic. In this article we shall give simple examples.

Number Symbols For Operations

Let 1 be "the operation of flipping an electric light switch in our home." Let 0 be "the operation of not flipping an electric light switch in our home." Let + stand for the words "followed by." Let = stand for "is equivalent to." Then, 1 + 1 = 0. Operation 1 followed by operation 1 is equivalent to operation 0. You may object that this is quite trivial and farfetched but you cannot assail its logic.

A small object is located in a fixed position *D* in the plane.



Let 0 be the operation of no rotation about *C*.
Let 1 be the operation of a counterclockwise rotation about *C* through 90°.
Let 2 be the operation of a counterclockwise rotation about *C* of 180°.
Let 3 be the operation of a counterclockwise rotation about *C* of 270°.
Let + be "followed by."
Let = be "is equivalent to."

1 + 2 = 3. Operation 1 followed by operation 2 is equivalent to operation 3.
3 + 3 = 2. Operation 3 followed by operation 3 is equivalent to operation 2.
3 + 1 = 0. Operation 3 followed by operation 1 is equivalent to no rotation about *C*.

The "addition" table below summarizes all operations.

+	1	2	3	0
1	2	3	0	1
2	3	0	1	2
3	0	1	2	3
0	1	2	3	0

To find the "sum" of 1 and 2. locate in the "addition" table the row headed 1 at the left side. Go across this row to the right until the column is reached which is headed 2 at the top. There the number 3 is found and this is the sum of 1 and 2.

This very same table also represents the addition table in modular arithmetic. modulo 4. (The subject of congruences is very important in number theory.) There we agree to add in the usual arithmetic manner the numbers 1,2,3, and 0., with the sole exception that every time the sum in arithmetic exceeds 3 we start again with 0,1,2, and 3. For example, 3 + 1 = 0. 3 + 2 = 1, etc. This may seem absurd but we do a similar thing every day of our life except that we count to 12 instead of 3. Just think of our system of telling time.

An Unusual Multiplication Table

We shall now use the four symbols, 1, −1, i and $-i$. The operation which combines these symbols will be called multiplication since it follows the usual rules of real algebra except that whenever i^2 appears, it is replaced by −1. The reader may recognize the i and the $-i$ as the "imaginary" numbers introduced in elementary algebra for the purpose of providing a solution for every quadratic equation. The multiplication table below summarizes all operations.

x	i	−1	$-i$	1
i	−1	$-i$	1	i
−1	$-i$	1	i	−1
$-i$	1	i	−1	$-i$
1	i	−1	$-i$	1

To find the product of i and −1, locate in the table the row headed i at the left side. Go across the row to the right until the column is reached which is headed −1 at the top. There the number $-i$ is found and this is the product of i and −1.

Every operation of multiplication yields one of the four symbols, i.e., we never get away from one of the four symbols.

If we substitue in this table + for x, 0 for 1, 1 for i, 2 for −1, and 3 for $-i$, we have exactly the addition table above. In other words, these two tables are fundamentally the same in this respect. There is a one-one correspondence between the symbols such that if 0 in the first table corresponds to 1 in the second table and two in the first table to −1 in the second table, then 2 + 0 in the first table corresponds to 1x(−1) in the second. Each table involves four symbols and there is one defined operation. An operation between two different symbols or between two like symbols slways yields one of the four symbols. There is a symbol in each table which has no effect on the other symbols when it combines with them, i.e. there is an operation which leaves everything unchanged. This symbol is called the identity symbol. It is 0 in the addition table and 1 in the multiplication table. Every symbol has an inverse, i.e., whatever one symbol does, there is another symbol that undoes it. Whatever operation you perform you can undo it. In other words, two unique symbols combine to form the identity symbol.

3 + 1 = 0	−1 x −1 = 1
2 + 2 = 0	1 x 1 = 1
0 + 0 = 0	i x $-i$ = 1

Operating on a symbol with 3 and then with 1 gives us back the original symbol since $3+1=0$. Similarly operating on a symbol with i and then with $-i$ gives us back the original symbol.

The associative law holds, i.e., we can group the symbols as we desire without changing the symbol which is the result of the operation, i.e., $1 + (2 + 3) = (1 + 2) + 3$. $(i \times -i) \times (-1) = i \times (-i \times -1)$.

In both tables the operation is commutative, i.e., $3 + 1 = 1 + 3$. $i \times -i = -i \times i$. We can change the order of two symbols in the operation without changing the symbol which is the result of the operation. In each case the four symbols constitute what mathematicians call a finite commutative or abelian group. The theory of groups is an important mathematical concept and is called by an eminent mathematician "the key to modern algebra and to modern geometry." (The concept that every algebraic equation is related to a group was first developed by Evariste Galois, a mathematical genius who was killed in a duel in 1832, at the age of 21. The story of his life reads like fiction and his untimely death reminds one of Moseley's fate at Gallipoli.)

Colors as Number Symbols.

For a moment let us look at the symbols 0, 1, 2, ..., 9 as the usual everyday numbers of arithmetic. We now propose to use colors instead of number symbols to represent values of a chosen unit of measurement. (There is no reason we cannot measure a deflection of a pointer by colors.) We are only interested in multiples of ten, i.e., the unit we shall use in our measurements is such that for our purpose values between 0 and 10 are of no consequence physically. 992 and 994 represent inconsequential differences from 990. We shall use a linear arrangement of three colors to represent a given value. The following table is set up.

Color	*Number*	*Color*	*Number*
Black	0	Green	5
Brown	1	Blue	6
Red	2	Violet	7
Orange	3	Gray	8
Yellow	4	White	9

For example, (red brown green) gives the value 215 in ordinary number symbols. From the table red is equivalient to 2, Brown to 1, and green to 5. But the third color is a multiplier. It does not stand for 5 but for multiplication by 1 followed by five zeros. So this linear arrangement of colors represents 2,100,000 or 21×10^5 units.

(brown green green) represents 1,500,000 units. Hence we can represent any value in multiples of ten from ten to 99,000,000,000. You may consider this system rather farfetched but actually it is in use today by radio engineers for use on resistors, the unit being the ohm. It is quite convenient to have colors since they stand out visually better than small numbers, and their value is not impaired by slight erasure. (Of course, you may facdtiously remark that the engineers must not be color blind.)

We Add to Multiply.

Sometimes in arithmetic the operation of addition seems to be principally multiplication. For example, the addition of the fractions 3/5 and 2/7 consists of four operations, three of which are multiplication. We multiply 3 by 7, and 2 by 5. The sum of these two products is the new numerator. The denominator of the sum is the product of 5 and 7. We now propose to let one number symbol have a one-one correspondence with another, and by doing so we shall substitute ordinary addition and subtraction for multiplication and division.

1	2	4	8	16	32	64	128	256	512	1024	2048	4096
0	1	2	3	4	5	6	7	8	9	10	11	12

The line is evenly spaced into twelve divisions. We start measuring from 0 through to 12. Above the line each number after the first is double the preceding number. The line together with numbers above and below can be similarly extended as far as desired. There is a one-one correspondence between the numbers above and below the line such that the arithmetical sum of any two counting numbers below the line corresponds to the product of the corresponding counting numbers above the line. $3 + 6 = 9$. $8 \times 64 = 512$. The arithmetical difference of any two counting numbers below the line corresponds to the division of the corresponding counting numbers above the line. $9 - 5 = 4$. $512 \div 32 = 16$. We interpret 0 and 1 as signifying that addition of 0 corresponds to multiplication by 1, i.e., 0 is the identity number for addition and 1 the identity number for multiplication. The scheme as outlined is too limited to be of practical use. We can draw up another scheme which is far more effective.

1	2	3	4	5	6	7	8	9	10
0	.3	.48	.6	.7	.78	.85	.9	.96	1

The numbers above the line are simply the positive integers in order. We cannot explain briefly how the numbers below the line are obtained but our purpose is merely to show that a one-one correspondence between number symbols can be set up so that addition in one case corresponds to multiplication in the other, and subtraction to division.

$.3 + .6 = .9 \qquad .96 - .48 = .48$

$2 \times 4 = 8 \qquad 9 \div 3 = 3$

Actually we can go beyond the scale as shown.

$1 + 1 = 2 \qquad .6 + 1 = 1.6$

$10 \times 10 = 100 \qquad 4 \times 10 = 40$

$2 + 1 = 3 \qquad .7 + .9 = 1.6$

$100 \times 10 = 1000 \qquad 5 \times 8 = 40$

(The reader may recognize that we have been using the principles of logarithms and slide rules.)

Calculating by Dozens Instead of Tens.

Let us consider our method of representing numbers greater than 9. For simplicity let us take the four digit number, 4302. Actually this is a condensed representation of $(4 \times 10^3) + (3 \times 10^2) + (0 \times 10^1) + (2 \times 10^0)$, where 10^0 is another symbol for unity. The representation 4302 is simply a linear arrangement of the multipliers of 10^3, of 10^2, of 10^1, and 10^0. Since the value of a digit in a number depends upon its position with respect to the other digits in the same number, the place occupied by a symbol is fully as important as the meaning of the symbol itself. For the same reason a zero multiplier cannot be omitted as it holds open a position in the display. This is quite obvious in such numbers as 10,000, etc.. In the number 111, due to position each 1 has a different meaning. When we multiply 2×6, we have no single symbol to represent the result and hence condense $1(10) + 2$ to 12. When in a multi-digit number we multiply 3×5 and "carry" 1, this "1" may be a multiplier of 10^1, 10^2, 10^3, etc.. We know which it is by the position of the 3 in the number being multiplied. We automatically place the 1 in the correct column. So long as a product of two integers is less than 10, a single symbol expresses the product but if the product is greater than 9, we must use our positional notation. It is quite clear that we ourselves have bestowed these special meanings upon the number symbols and in fact it took many centuries before the present result was attained. If you are in doubt as to its efficiency, try multiplying in Roman numerals.

We are so accustomed from childhood to this decimal representation that we entirely forget that not only are other systems possible but that history tells us that they have been used in the past by various different cultures. Let us consider how we might be writing our numbers today if nature had given us six fingers on each hand. In fact, those cultures who used the vigesimal (counting by twenties) system apparently continued their counting from ten on to 20 by using their toes. A duodecimal representation means that we count by twelves, i.e., we substitute 12 for 10 so that we have $(a \times 12^3) + (b \times 12^2) + (c \times 12^1) + (d \times 12^0)$ for a four digit number *abcd* in this system. However *a* is a multiplier of 12^3, *b* of 12^2, *c* of 12^1, and *d* of 12^0.

We must have twelve symbols and *a*, *b*, *c*, and *d* can be anyone of them. We can use our usual symbols from 0 to 9 for ten of these symbols. Then we invent Ø for 10 and ϕ for 11. When *d* for example exceeds ϕ, we use a two digit number.

Decimal System	*Duodecimal System*
10 = 0(12) + 10	Ø
11 = 0(12) + 11	ϕ
12 = 1(12) + 0	10
13 = 1(12) + 1	11
14 = 1(12) + 2	12
18 = 1(12) + 6	16
20 = 1(12) + 8	18
22 = 1(12) + 10	1Ø
44 = 3(12) + 8	38
60 = 5(12) + 0	50
144 = 1(122) + 0(12) + 0	100
70 = (5)(12) + 10	5Ø
71 = (5)(12) + 11	5ϕ

2 × 2 = 4. 3 × 3 = 9.
2 × 4 = 8. 2 × 5 = Ø
3 × 4 = 10. 2 × 6 = 10.
2 × ϕ = 1Ø.
2 × 10 = 38.
4 × 9 = 30.
2 × 50 = Ø0.
2 × 60 = 100.

Fractions

Decimal System		*Duodecimal System*
1/12	.0833...	0.1
2/12	.1666...	0.2
	.25	0.3
	.3333...	0.4
	.41666...	0.5
	.5	0.6
11/12		0.ϕ

To convert 1/10 in the decimal system to the duodecimal system:

$$1/10 - 1/12 = 1/60 = 1/(5)(12)$$

$$(1/(5)(12) = \frac{12/5}{12^2} = \frac{2}{12^2} + \frac{2/5}{12^2}$$

$$\frac{2/5}{12^2} = \frac{24/5}{12^3} = \frac{4}{12^3} + \frac{4/5}{12^3}$$

$$\frac{4/5}{12^3} = \frac{48/5}{12^4} = \frac{9}{12^4} + \frac{3/5}{12^4}$$

The complete duodecimal representation of the decimal fraction 1/10 is $0.1\dot{2}49\dot{7}$.

Today there are people who insist that a duodecimal system would be better than our decimal system. Since 12 can be equivalently written as (2)(6), (3)(4), and (3)(2)(2), while 10 can be written only as a product of the prime factors, 2 and 5, more common fractions would be terminating or finite in this duodecimal system. This is shown above.

If you were buying 20 eggs in the duodecimal system you would expect to receive two dozen. An object 10 inches long would be one foot in length. 100 square inches =1 square foot. 1000 cubic inches = 1 cubic foot. 1 foot = 0.4 yard.

By changing the meaning of the symbols and leaving the operations unaltered we arrive at results which differ from ordinary arithmetic but which are just as valid. In the United States some of our common units of measurement are certainly hybrid in origin. With their pounds, guineas, shillings and pence, the sterling countries are at least more consistent in their confusion.

Milford, Texas.

CERTAIN TOPICS RELATED TO CONSTRUCTIONS WITH STRAIGHTEDGE AND COMPASSES

(semi-popular)

Adrien L. Hess

Introduction

Closely related to the problem of geometric constructions are certain topics which serve to extend and enrich the usual conception of such constructions. The topics represent various facets of the problem which have been developed within the last one hundred sixty years. Of such topics, the three most closely related to geometric constructions are: Geometrography, Paper-folding and Match Stick Geometry.

Geometrography

In 1833 Steiner (17), an outstanding German mathematician, suggested that every construction in geometry should be studied so that the solution used would be the simplest, the most exact, and the surest. He also proposed that this study should include constructions in general, and constructions made under limitations as to instruments used and with obstructions existing in the plane. Nothing seems to have materialized from the outlining of this problem until, in 1884, Wiener solved several constructions for which he counted the number of circles and straight lines drawn (2).

Lemoine, who made the first systematic approach to the problem, presented his initial ideas to the leading French scientific society of his time in 1888. In less than fifteen years he wrote more than thirty notes and memoirs, which appeared in many mathematical and scientific journals, in which he amplified and extended his ideas on geometrography. Starting in 1888 with geometrography as applied to straightedge and compasses construction, he had extended his system by 1894 to include descriptive geometry and by 1902 to include geometry of three dimensions (9; 10). In 1902 Lemoine summed up his development of geometrography in his book *Geometrographie, ou Arts des Constructiones Geometriques* (11).

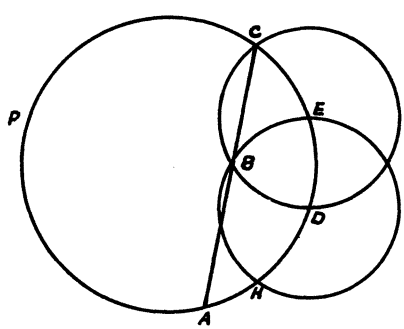
Although geometrography was developed mainly by Lemoine, its growth was aided by the contributions and comments of many writers in England, France and Germany. Other systems of geometrography were devised by Papperitz (13) in 1908 and by Grüttner (8) in 1909. Lemoine, Godeaux (7) and Adler (1) extended the system to include tools other than the straightedge and compasses. In 1929 Tuckey (18) devised a system of geometrography in which he considered only the settings of the straightedge or compasses and the number of straight lines or circles drawn. Some recent college geometry textbooks (4; 16) include a brief discussion of geometrography.

Lemoine chose two operations for the straightedge and three operations for the compasses as the fundamental operation in his system of geometrography. The five operations and their symbols are:

(1)	To place the straightedge on a given point	R_1
(2)	To draw a straight line with a straightedge	R_2
(3)	To place one point of the compasses on a given point	C_1
(4)	To place one point of the compasses on any point of a line	C_2
(5)	To draw a circle	C_3

If, in a construction, these operations occur respectively a_1, a_2, b_1, b_2, b_3 times, the symbol for the construction is $a_1R_1 + a_2R_2 + b_1C_1 + b_2C_2 + b_3C_3$. The total number of operations is the sum $a_1 + a_2 + b_1 + b_2 + b_3$, which is called the coefficient of simplicity (S). The sum $a_1 + b_1 + b_2$ is the total number of coincidences and is called the coefficient of exactitude (E).

The system will be illustrated by Swales' Construction for finding the radius of a circle when the center O is not given.



With any point O on the given circle $O(P)$ and any convenient radius r, draw circle $D(r)$ to intersect $O(P)$ at H and E.	$C_2 + C_3$
With E as center draw circle $E(r)$ to intersect $O(P)$ at C and $D(r)$ at B.	$C_1 + C_3$
Draw the straight line BC to intersect $O(P)$ at A. AB is the radius of the circle $O(P)$	$2R_1 + R_2$

The symbol for the entire construction is $2R_1 + R_2 + C_1 + C_2 + 2C_3$. The coefficient of simplicity is $S = 2 + 1 + 1 + 1 + 2 = 7$. The coefficient of exactitude is $E = 1 + 1 + 2 + 4$. The construction with the smaller coefficient of simplicity is considered the simpler construction.

Paper Folding

Although, historically, the folding of materials and the making of knots are quite old, their application to geometry has been made in more recent times. It was about five hundred years ago that the great German, Albrecht Dürer, who was interested in geometry as well as art, first showed that the regular and semi-regular solids could be constructed out of paper by marking the boundaries of the polygons, all in one piece, and then folding the polygons along the connected edges (3). The first English translation of Euclid's *Elements*, printed in 1570, included a most interesting feature. In the eleventh Book of the translation, figures made of paper were pasted in such a way that they could be opened up to make actual models of space figures (14). Over a century later Urbano D'Aviso, a student of Cavalieri, published a work in Rome entitled *Trate de la Sphere*, in which geometric constructions were worked out by means of paper folding. The formation of a regular hexagon and a regular pentagon by means of knots, a type of paper folding, is attributed to him (6).

In 1893 two men of different nationalities and in widely separated countries wrote works on paper folding. Wiener, a teacher in a German polytechnic school, showed how to construct regular convex polyhedra by paper folding (2). Row, a mathematician of India, wrote a book in which he gave a more complete treatment of paper folding (14). This work was translated by Beman and Smith in 1901 and the book became readily available in this country.

In 1905 another book, entitled *First Book of Geometry*, appeared which used paper folding. The authors Grace C. Young and W. H. Young, feeling that Row's book was too advanced for children and too puerile for adults, wrote their book to meet the needs of children. In 1908 it was translated into German under the title *Der Kleine Geometer* (20). The book is designed to give instruction to young children in fundamental ideas of plane and solid geometry. No particular apparatus is needed for the constructions chosen and these constructions can be made and understood by children four and five years of age. Besides the usual fundamental constructions of geometry, other constructions are given in the book to develop understanding of the concept of inequality, regular polygons, parallel lines and planes, and the theorem of Pythagoras.

As shown by Yates (19) with properly chosen postulates, all constructions of plane geometry that can be carried out with a straightedge and compasses can be executed by paper folding.

Match Stick Geometry

Match stick geometry, devised by Dawson (5) in 1939, uses as its sole tool a finite supply of match sticks of equal length. For his geometry he chose four postulates:

(1) A straight line may be laid to pass through a given point, or with one extremity on a given point.

(2) A line may be laid to pass through two given points, or with one extremity at one given point and passing through a second point, but the two points may not be such as lie in a given line or laid line.

(3) A line may be laid with one extremity at a given point and its other extremity on a given line.

(4) Two lines may be laid simultaneously to form the sides of an isosceles triangle, two of their extremities coinciding and the other two being given points.

Two lemmas and an assumption complete the geometry. The lemmas are: A given line of a length less than, equal to, or greater than the length of a match stick can be bisected; a line can be laid through a given point and parallel to a given line. Since a circle cannot be drawn, it is assumed that a circle is determined when its center and a point on the circumference are given.

The construction of a half hexagon is characteristic of the operations of this geometry. The equilateral triangle ABC is constructed. On side BC the equilateral triangle BCD is constructed with D distinct from A. On side DB the equilateral triangle DBE is constructed to form the half hexagon $ACDE$. Thus AB is extended in a straight line so that $AE = 2AB$. This construction also gives a way of constructing a line parallel to a given line, for CD is parallel to AB.

Under the postulates and the assumptions stated above, it is possible to perform all constructions which are possible with a straightedge and compasses.

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PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

264. *Proposed by Norman Anning, Alhambra, California.*

Four points are thrown at random on a plane. What is the probability that they will be the vertices of a convex quadrilateral?

265. *Proposed by Stephen Armstrong, Union College Schenectady, New York,*

What is the volume of revolution of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ revolved about the line $x = y$?

266. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

If M and M' are points inverse to each other with respect to the circumcircle of a triangle ABC, then prove that:

$$\angle BMC + \angle BM'C = 2\angle A$$
$$\angle CMA + \angle CM'A = 2\angle B$$
$$\angle AMB + \angle AM'B = 2\angle C$$

267. *Proposed by Alan Wayne, Cooper Union School of Engineering, New York.*

If $R_m = \sum_{k=1}^{m} \frac{\sqrt{k^n - (k-1)^n}}{\sqrt{m^{n+1}}}$, prove that $\lim_{m\to\infty} R_m = \frac{2\sqrt{n}}{n+1}$

268. *Proposed by J. W. Clawson, Collegeville, Pennsylvania.*

Three coaxal circles, centers at A, B and P have the common points C and D. Any straight line is drawn through C cutting the circles again in the points L, M and N respectively. Prove that the ratio LN/NM equals the ratio AP/PB.

269. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn.*

Find the sum $\sum_{n=1}^{\infty} \left[\frac{n}{1!} + \frac{n-1}{2!} + \frac{n-2}{3!} + \dots + \frac{1}{n!} \right] x^n$

270. *Proposed by Leon Bankoff, Los Angeles, California.*

A maximum circle is inscribed in a crescent formed by a semicircle and a quadrant of a circle. Find a general expression for the radii of consecutively tangent circles touching the sides of the crescent, the first touching the maximum circle, the second touching the first and so on.

SOLUTIONS

Late Solutions

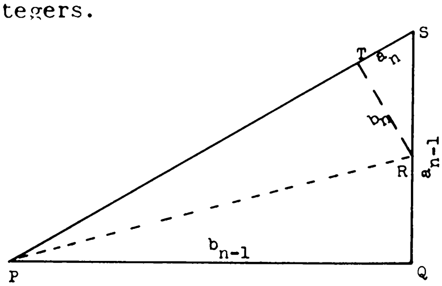
229, 230, 231, 232, 234 *(partially)* **235** *Richard K. Guy, University of Malaya, Singapore.*

242. *Thomas F. Mulcrone, St. Charles College, Grand Coteau, Louisiana.*

Irrationality of $\sqrt{3}$

243. [September 1955] *Proposed by William R. Ransom, Tufts College.*

In X:2 Euclid proves that the secant of 45° cannot be expressed as the ratio of integers. Using this method of proof, bisecting the angle and infinite descent, show that the tangent of 30° cannot be expressed as the ratio of integers.



1. *Solution by the proposer.* SPQ is a 30° right triangle with R on the bisector of $\angle SPQ$, and RT perpendicular to PS. Then we have

$$a_n = ST = PS - PT = 2QS - PQ = 2a_{n-1} - 2b_{n-1}$$

$$b_n = RT = QR = QS - RS = QS - 2ST = a_{n-1} - 2a_n$$

Hence if $\tan 30° = a_{n-1}/b_{n-1}$ is the ratio of integers, we can get a smaller triangle in which $\tan 30° = a_n/b_n$, also the ratio of integers. But if to the members of the b_n equation we add the members of the inequality $a_n < b_n$ we get $a_n + b_n < a_{n-1} - 2a_n + b_n$ whence $a_n < 1/3\, a_{n-1}$ It follows that a_n is less than $a_0 \div 3^n$, and so it cannot be an integer for all values of n. This contradiction shows that a_0/b_0 could not have been the ratio of integers.

II. *Solution by Richard K. Guy, University of Malaya, Singapore.* The following method does not use bisection of the angle, however it does use infinite descent and is essentially equivalent to the proof of the irrationality of $\sqrt{2}$ found in Hardy. Pure Mathematics, 6th Ed., 1933, p6. Let ABC be a 30°, 60° right triangle with $\angle A = 30°$. Angles of 45° and 60° are also constructed on BA at A whose sides meet BC produced D and E. The perpendicular to EC at D meets AE at F. It is easily shown that if $AB = a$ and $BC = b$ then $DF = CD = a - b$ and $DE = 3b - a$. Thus if $\tan 30° = \frac{1}{\sqrt{3}} = \frac{b}{a}$ then also $\tan 30° = \frac{a-b}{3b-a}$. The triangle EDF is smaller than ABC, yet has commensurable sides if ABC has. Repetition of this construction leads to the required contradiction.

Norman Anning pointed out that this problem appears on p. 175 of W. R. Ransom's book 100 Mathematical Curiosities, 1955, Published by J. W. Walch, Portland, Maine.

Equivalence of Sums

244. [September 1955] *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Take any nine consecutive positive integers and find among them (with only three duplications) two sets of six integers such that their sums, the sums of the squares and the sums of their cubes are equal.

Solution by H. R. Leifer, Pittsburgh, Pennsylvania. This is a special case of the Tarry-Escott problem. If $a_1, a_2, \ldots a_s \stackrel{k}{=} b_1, b_2, \ldots b_s$ represents two sets of integers equal in number such that the integers in each set have the same sum, the same sum of squares, etc. up to and including the same sum of the kth powers, then

$$a_1, a_2, \ldots a_s, b_1+h, b_2+h, \ldots b_s+h \stackrel{k+1}{=} b_1, b_2, \ldots b_s, a_1+h, a_2+h, \ldots a_s+h$$

Where h is any arbitrary number.
Let $a_0, a_1, \ldots a_8$ represent the nine consecutive positive integers. To meet the conditions of the problem a_0 should be included in one of the initial sets and suitable h's should be selected so that no integer exceeding a_8 occurs in the final set. $a_0, a_3 \stackrel{1}{=} a_1, a_2$ then $a_0, a_3, a_3, a_4 \stackrel{2}{=} a_1, a_2, a_2, a_5$; $h = 2$ and $a_0, a_3, a_4, a_4, a_5, a_8 \stackrel{3}{=} a_1, a_2, a_2, a_6, a_6, a_7$; $h = 3$ which is the desired solution.

(Use of alternate initial sets and other h's will give the same solution)

Also solved by Richard K. Guy, University of Malaya, Singapore; Chih-yi Wang, University of Minnesota and the proposer.

Direct Integration

245.[September 1955] *Proposed by Chih-yi Wang, University of Minnesota.*

Evaluate $\int_0^{\pi/4} \sqrt{\tan x}\, dx$ without using approximation methods.

Solution by C. F. Pinzka, Princeton, New Jersey. Letting $y = \sqrt{\tan x}$, the given integral becomes

$$\int_0^1 \frac{2y^2 dy}{1+y^4} = \frac{1}{\sqrt{2}} \int_0^1 \left[\frac{y}{y^2 - \sqrt{2}\,y + 1} - \frac{y}{y^2 + \sqrt{2}\,y + 1} \right] dy$$

$$= \frac{1}{2\sqrt{2}} \log \frac{y^2 - 2y + 1}{y^2 + \sqrt{2}\,y + 1} + \frac{1}{\sqrt{2}} \text{arc tan} \frac{\sqrt{2}\,y}{1 - y^2} \Bigg|_0^1$$

$$= \frac{1}{\sqrt{2}} \left[\log(\sqrt{2} - 1) + \frac{\pi}{2} \right]$$

Remark: In a sense this is still not the solution required, since π and $\log(\sqrt{2} - 1)$ must still be evaluated approximately. One could just as well have given a name such as Ugh(x) to the series resulting from direct integration of $y^2 - y^6 + \dots$.

Also solved by PFC Billy J. Boyer, Murphy Army Hospital, Waltham, Massachusetts; Huseyin Demir, Zonguldak, Turkey; H. M. Feldman, Washington University, St. Louis, Missouri; Richard K. Guy, University of Malaya, Singapore; Ratib A. Karam, University of Florida; Joseph D. E. Konhauser, State College, Pennsylvania; Walter D. Lambert Canaan, Connecticut; Carman E. Miller, University of Saskatchewan; Lawrence A. Ringenberg, Eastern Illinois State College; Hazel S. Wilson, Jacksonville State Teachers College, Alabama and the proposer.

Maclaurin's Expansion

246. [September 1955] *Proposed by A.S.Gregory, University of Illinois.*

Let a function be defined by $b\, f(x) - ax = \sin(a\, f(x) + bx)$ with $0 < a < b$, $a^2 + b^2 = 1$. Expand $f(x)$ in a Maclaurin's Series.

Solution by Chih-yi Wang, University of Minnesota. Let $w = af(x) + b$. Then the given function is transformed, by aid of $a^2 + b^2 = 1$ into $F(x,w) = 0$, where

$$F(x,w) = x - bw + a \sin w$$

$$(1) \qquad = x - (b - a)\, w - \frac{aw^3}{3!} + \frac{aw^5}{5!} - \frac{aw^7}{7!} + \frac{aw^9}{9!} - \cdots .$$

Since $F(0,0) = 0$, $F_w(0,0) = -(b-a) \neq 0$, and

$$|\text{any coefficient of } w \text{ in } F(x,w)| < 1,$$

it is well known that the Maclaurin's series of w, hence $f(x)$, exists and represents an analytic function in the neighborhood of $x = 0$. The majorant series of $w(x)$ is

$$w = \tfrac{1}{4}\,[1 - (1-x)^{-\frac{1}{2}}(1-9x)^{\frac{1}{2}}]$$

By substituting $w = c_1x + c_2x^2 + c_3x^3 + \ldots$ into (1), and equating the coefficients of x successively we obtain $c_1 = 1/(b-a)$,

$$c_2 = 0,\ c_3 = -a/3!(b-a)^4,\ c_4 = 0,\ c_5 = a(9a+b)/5!(b-a)^7,$$

$$c_6 = 0,\ c_7 = -a(225a^2 + 54ab + b^2)/7!(b-a)^{10},\ c_8 = 0,$$

$$c_9 = a(11025a^3 + 4131a^2b + 243ab^2 + b^3)/9!(b-a)^{13}, \ldots$$

Hence, by aid of $a^2 + b^2 = 1$, we get

$$f(x) = \frac{b+a}{b-a}x - \frac{x^3}{3!(b-a)^4} + \frac{(9a+b)x^5}{5!(b-a)^7} - \frac{(225a^2 + 54ab + b^2)x^7}{7!(b-a)^{10}}$$

$$- \frac{(11025a^3 + 4131a^2b + 243ab^2 + b^3)x^9}{9!(b-a)^{13}} - \ldots$$

Note that the method of undetermined coefficients can be applied to the following well known more general problem:

Let the following relation be given

$$F(z,w) = \sum_{h=0}^{\infty} A_h(z)\,w^h = 0$$

where

$$A_h(z) = \sum_{k=0}^{\infty} a_{hk} z^k; \quad h = 0, 1, 2, \ldots$$

such that $|a_{hk}| < M$; $h, k = 0, 1, 2, \ldots$ (M is independent of h and k),

$$F(0,0) = a_{00} = 0, \quad F_w(0,0) = a_{10} \neq 0.$$

Then we can expand w in the neighborhood of $z = 0$ to the following form

$$w = c_1z + c_2z^2 + c_3z^3 + \ldots$$

which represents an analytic function, whose majorant series is

$$w = \frac{1}{2(1+M)} \left[1 - (1-z)^{-\frac{1}{2}} \{ 1 - (1+2M)^2 z \}^{\frac{1}{2}} \right].$$

The given problem is a very special case of the general problem.

Also solved by Richard K. Guy, University of Malaya, Singapore.

A Greatest Integer Function

247. [September 1955] *Proposed by Julian H. Braun, White Sands Proving Ground, New Mexico.*

Reduce $f(n) = \sum_{k=1}^{[3n/4]} \sum_{i=1}^{u} 3i$ to the form $[g(n)]$ where $u = n - [(4k-1)/3]$, $g(n)$ is a polynomial and $[x]$ denotes the greatest integer less than or equal to x.

I. *Solution by the proposer.*

$$f(n+4) = \sum_{k=1}^{[3n/4+3]} \sum_{i=1}^{u+4} 3i,$$

$$= \sum_{k=-2}^{[3n/4]} \sum_{i=1}^{u} 3i,$$

$$= \sum_{i=1}^{n+3} 3i + \sum_{i=1}^{n+2} 3i + \sum_{i=1}^{n+1} 3i + f(n);$$

$$f(n+4) - f(n) = 3(3n^2 + 15n + 20)/2.$$

Now let $f(n) = [an^3 + bn^2 + cn + d]$. Then

$f(n+4) - f(n) = 12an^2 + (48a + 8b)n + 64a + 16b + 4c$

$12a = 9/2$; $a = 3/8$. $48a + 8b = 45/2$; $b = 9/16$.

$64a + 16b + 4c = 30$; $c = -3/4$. Therefore

$f(n) = [(3/16)n(2n + 3n - 4) + d]$. We find

$f(0) = 0 = [d]$, $f(1) = 0 = [3/16 + d]$, $f(2) = 3 = [15/4 + d]$,

$f(3) = 12 = [207/16 + d]$, whence $0 \le d < 1/16$.

The simplest form for $g(n)$ is obtained by setting $d = 0$.

II. *Heuristic treatment by Richard K. Guy, University of Malaya, Singapore.* To develop a formula for $g(n)$ we calculate the first few values, tabulate, then find the first, second and third differences. The third differences recur and average 9/4. The leading term is therefore $9/4 \cdot n^3/6$ or $3/8 n^3$. This can be obtained by observing that the sum is three times the number of points in three dimensional space with integral coordinates inside a tetrahedron whose perpendicular edges are about n, n, $(3n)/4$. That is:

$$3 \cdot 1/3 \cdot 1/2\, n^2 \cdot 3n/4 = 3/8 n^3.$$

Next calculate the difference between the actual values and $3n^3/8$. These have their second differences recurring and averaging 9/8. This gives a second term of $9/8 \cdot n^2/2 = (9n^2)/16$. Continuing in this fashion the third term is $-3n/4$ and the fourth term is 0, $-3/16$, $-3/4$ or $-15/16$ according as $n = 0, 1, 2, 3 \pmod 4$. These all lie between zero and -1 so the result $g(n) = (3n)/16\,(2n^2 + 3n - 4)$ follows.

Also solved by Chih-yi Wang, University of Minnesota.

Collinear Points

248. [September 1955] *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Let Γ_1 and Γ_2 be two plane curves. Let t be a variable line intersecting these curves at the points M_1, M_2 where the tangents t_1 and t_2 to the curves are parallel to each other. Prove that the centers of curvature C_1 and C_2 of Γ_1 and Γ_2 at M_1 and M_2 are collinear with the characteristic point C of the straight line t.

Solution by the proposer. Considering the new position t' of t very close to t, we have $M_1M_1'/\sin \Delta\theta = C'M_1'/\sin M_1$, where M_1' is close to M_1 on Γ_1, and $\Delta\theta = (t, t')$; the angle between t and t'.

Infinitesimally

$$ds_1/d\theta = CM_1/\sin \mu_1, \qquad \mu_1 = (t, t_1) \pm \pi$$

and similarly

$$ds_2/d\theta = CM_2/\sin \mu_2, \qquad \mu_2 = (t, t_2) \pm \pi$$

Having $\sin \mu_1 = \sin \mu_2$, as t_1 is parallel to t_2, we get

$$ds_1/CM_1 = ds_2/CM_2$$

which in turn yields

$$(ds_1/d\alpha)CM_1 = (ds_2/d\alpha)CM_2$$

i.e.

$$R_1/CM_1 = R_2/CM_2$$

where $d\alpha$ is the infinitesimal angle relative to the parallel normals at M_1, M_2, and R_1, R_2 the corresponding radii of curvature. The last equality proves the statement.

Also solved by Richard K. Guy, University of Malaya, Singapore and Chih-yi Wang, University of Minnesota.

A Polynomial of Degree n^2

249. [September 1955] *Proposed by David Sayre, IBM Corp., New York.*

The zeros of a polynomial $P(z)$, of degree n^2, all lie on the unit circle and are expressable as exp $i(a_j - a_k)$ with $j, k = 1,2,3,\ldots n$. The a's are a set of n unknown real numbers. What can be said about the coefficients of the polynomial?

Solution by Chih-yi Wang, University of Minnesota. Let $P(z)$ be written in the following form:

$$P(z) = K\left[z^{n^2} + \sum_{r=1}^{n^2}(-1)^r C_r z^{n^2-r}\right], \quad K \neq 0$$

Then we can say that all C_r are real symmetric functions of a's furthermore $C_{n^2} = 1$, $C_r = C_{n^2-r}$ for $r = 1, 2, \ldots, n^2-1$, for $P(z)$ can also be written in the factored form:

$$P(z) = K(z-1)^n \prod_{s>t} [(z^2+1) - 2z\cos(a_s - a_t)]$$

where s runs from 2 to n^2 and t runs from 1 to $n^2 - 1$. The last property of C_r mentioned above follows from the fact the $P(z)$ is a reciprocal equation.

Also solved by Richard K. Guy, University of Malaya, Singapore.

Guy pointed out that the coefficients could be expressed in the following form:

For $n = 1$: $x - 1$

$n = 2$: $(x-1)^2\,[x^2 - 2x\cos(a_1 - a_2) + 1]$

and in general

$$(x-1)^n\,[x^{n^2-n} - 2x^{n^2-n-1}\sum_{i>j}\cos(a_i - a_j) + \ldots + 1]$$

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and source, if known.

Q 165. The rectangular axes x and y are rotated through an angle α. If the new axes are designated as X and Y, find $\frac{dY}{dX}$ and $\frac{d^2Y}{dX^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [*Submitted by Henry E. Fettis.*]

Q 166. Can the sum of the cubes of the first m consecutive integers equal the sum of the cubes of the next n consecutive integers? [*Submitted by Huseyin Demir.*]

Q 167. Prove that $1 + 1/3 + 1/5 + \ldots + 1/2n - 1$ for $n > 1$ can never be an integer. [*Submitted by M. S. Klamkin.*]

Q 168. Evaluate the determinant

$$\Delta = \begin{vmatrix} a^2 + d^2 & ab + de & ac + df \\ ab + de & b^2 + e^2 & bc + ef \\ ac + df & bc + ef & c^2 + f^2 \end{vmatrix}$$

[*Submitted by Selig Starr*].

Q 169. Show that if u, v are odd and $p > 0$, then

$$\frac{(u^{2^p m} - v^{2^p m})}{(u^{2m} - v^{2n})} = 2^{p-1}\,(2M - 1),$$ M and integer.

[*Submitted by Richard C. Bartell.*]

ANSWERS

A 165. If the tangent to an arbitrary curve makes angles θ and γ with the x and X axes respectively then $\tan\gamma = \dfrac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$ or

$$\frac{dY}{dX} = \frac{\cos\alpha\dfrac{dy}{dx} - \sin\alpha}{\sin\alpha\dfrac{dy}{dx} + \cos\alpha}$$

Next the curvature K is unaffected by the change in coordinate systems. Hence,

$$K = \frac{\dfrac{d^2Y}{dX^2}}{\left[1 + \left(\dfrac{dY}{dX}\right)^2\right]^{3/2}} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

or

$$\frac{d^2Y}{dX^2} = \frac{d^2y}{dx^2}\left\{\frac{1}{\left(\sin\alpha\dfrac{dy}{dx} + \cos\alpha\right)^3}\right\}$$

Relations between higher derivatives can be found by equating $\dfrac{dK}{ds}$, $\dfrac{d^2K}{ds^2}$ etc.

A 166. If $1^3 + 2^3 + \ldots + m^3 = (m+1)^3 + \ldots + (m+n)^3$

then $2(1^3 + 2^3 + \ldots + m^3) = 1^3 + 2^3 + \ldots + (m+n)^3$

or $2[\frac{1}{2}m(m+1)]^2 = [\frac{1}{2}(m+n)(m+n+1)]^2$

so $\frac{1}{2}m(m+1)\sqrt{2} = \frac{1}{2}(m+n)(m+n+1)$

But this last equation is impossible. Therefore the answer is no.

A167. Assume the sum S is integral and let p be the greatest prime less than or equal to $2n-1$. There will not be any other multiples of $p < 2n-1$, since between a and $2a$ there lies a prime. Then

$$1 + 1/3 + 1/5 + \ldots + 1/2n - 1 - 1/p = s - 1/p = \frac{ps-1}{p}$$

But the right side has a denominator p while the left side does not. This is a contradiction which proves the original statement.

A 168. It is clear that Δ is the square of the determinant

$$\begin{vmatrix} a & d & 0 \\ b & e & 0 \\ c & f & 0 \end{vmatrix}$$ which is zero.

A 169.

$$u^{2^p m} - v^{2^p n} = (u^m)^{2^p} - (v^n)^{2^p}$$

$$= [(u^m)^{2^{p-1}} + (v^n)^{2^{p-1}}][(u^m)^{2^{p-2}} + (v^n)^{2^{p-2}}]\ldots$$

$$[(u^m)^2 + (v^n)^2][u^{2m} - v^{2n}].$$

Hence the quotient contains $p-1$ factors. Since u, v are odd, each is of the form $2k-1$, and $(u^m)^{2q}, (v^n)^{2q}$ are each of the form $4k+1$. Hence each of the $p-1$ factors of Q is of the form $2(4k+1)$. It follows that the quotient equals $2^{p-1}(4k+1)$ or $2^{p-1}(2M-1)$.

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 21. Given a standard 8 x 8 chess board. Remove the two diagonally opposite squares. Can this altered board be covered by 31 tiles, each tile of dimension 1 x 2 squares, without gap or overlap?

[*Submitted by R.F.Benton*]

T 22. Solve the simultaneous system

$$\cos A \cos B + \sin A \sin B \sin C = 1$$

$$\sin A + \sin B = 1$$

$$A + B + C = 180°$$

[*Submitted by M.S.Klamkin*]

T 23. In France during the war when tobacco was scarce, it was customary to save cigarette butts and roll new cigarettes out of their tobacco. If one cigarette can be rolled from three butts, what is the total number of cigarettes one can get from an ordinary pack of 20? [*Submitted by Paul B. Johnson.*]

SOLUTIONS

S 21. No. Removing the two diagonally opposite corners removes two squares of the same color, while each 1 x 2 tile must cover two squares of different color.

S 22. From the first and third equation it follows that $\cos(A - B) \geq 1$. Thus $A = B$ and then $C = 90$. But this does not satisfy the second equation. Therefore the equations are inconsistent.

S 23. One can get 30 cigarettes!! The first 20 yield 20 butts from which 6 cigarettes are rolled with two spare butts. The six new butts yield two cigarettes. From four butts we make the 29th cigarette. With the two tired butts in hand, we borrow a butt from a kind friend, with which we make our 30th cigarette.

But what about this borrowed butt, you say? Simple. We smoke the thirtieth cigarette, return the butt and everything is even!

OUR CONTRIBUTORS

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